Drift-dominated conduction within an Ohmic medium

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Drift-dominated charge transport within an Ohmic medium is analyzed using the method of characteristics to convert the governing partial differential equations into a set of ordinary differential equations. As for earlier work within a lossless medium, the analysis is generalized to allow any initial and boundary conditions and any terminal constraint or excitation. For imposed currents the equations are exactly integrable, while for imposed voltages, the equations are easily integrated using the Runge-Kutta method of numerical integration. Special cases examined are the charging transients for space-charge-limited conduction for step current and step voltage excitations from rest and the discharging transients for systems in the dc steady state for which the excitation is instantaneously open or short circuited.

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I. INTRODUCTION

Drift-dominated conduction models are used in semiconductors,1,3,4 dielectrics,3,5-7 and electrets8 where the origin of space charge can be due to any number of mechanisms. In many materials irradiation by an electron beam,8 light,9 or nuclear radiation10 generates charge carriers. Charge injection also occurs from electrodes under high field strengths either by field emission or by contact charging of impurities. However, each of these different charge-injection mechanisms are analytically represented by various initial and boundary conditions. The subsequent charge transport is independent of the actual mechanism of how the charge is generated. Most past work9-11 concerned with charge transport assume the surrounding medium is lossless. Here we take into account the material’s natural Ohmic conduction so that any injected space charge is neutralized as it travels.11 As for the lossless case,1,9-11 the method of characteristics is used to convert the governing partial differential equations into a set of first-order ordinary differential equations which are easily numerically integrated by the Runge-Kutta method. For current source excitations closed-form results are obtained. As in past work, the mathematical development will allow any initial and boundary conditions and any terminal constraint or excitation. In particular we examine the charging transients for space-charge-limited conduction for step current and step voltage excitations from rest and the discharging transients for systems in the dc steady state for which the excitation is instantaneously open or short circuited.

II. FIELD EQUATIONS

We consider a parallel-plate geometry of spacing l where the lower positive electrode at \( x = 0 \) is a source of ions with mobility \( \mu \) in the dielectric medium of permittivity \( \varepsilon \) and Ohmic conductivity \( \sigma \). We neglect fringing and consider only one-dimensional variations with the coordinate \( x \). Without loss of generality we assume the injected charge to be positive, so that the governing equations are

\[
\begin{align*}
\nabla \times \mathbf{E} &= 0, \quad \int \mathbf{E} \, dx = q, \\
\nabla \cdot (\varepsilon \mathbf{E}) &= q, \\
\n\nabla \cdot \mathbf{J}_e &= \frac{\partial q}{\partial t}, \\
\n\mathbf{J}_e &= q \mu \mathbf{E} + \sigma \mathbf{E}.
\end{align*}
\]

The charge density \( q \) is only due to the injected charge and has no contribution from the Ohmic charge carriers. In the Ohmic conduction model one species of charge (electrons) moves relative to a stationary background species (positively charged nucleus) with a drift velocity proportional to the electric field. Thus even though the net charge for Ohmic conduction is zero a net Ohmic conduction current flows. Further charge injected into the medium with mobility \( \mu \) is not canceled out by any background charge. The transient solution presented here is analogous to other work with intrinsic relaxation semiconductors12 in the limit where the conductivity \( \sigma \) is a constant with diffusion and trapping negligible. As the injected charge drifts it is neutralized by the Ohmic relaxation process. The important time constants are the charge transport time compared to the charge relaxation time \( \varepsilon / \sigma \).

For voltage source excitations it is convenient to introduce the nondimensional variables

\[
\begin{align*}
\tilde{E} &= \frac{E}{l}, \quad \tilde{q} &= \frac{q}{l^2 \mu \varepsilon}, \quad \tilde{J} = \frac{J}{l \mu \varepsilon}, \\
\tilde{x} &= \frac{x}{l}, \quad \tilde{t} = l \varepsilon / \mu \varepsilon, \quad \tilde{\tau} = \frac{l \mu \varepsilon}{\mu \varepsilon}.
\end{align*}
\]

For current source excitations it is useful to use as nondimensional variables

\[
\begin{align*}
\tilde{E} = \frac{(\varepsilon \mu / l^2)}{l^2}, \quad \tilde{q} = \frac{(\mu v / l)}{l^2}, \quad \tilde{J} = \frac{(\mu \varepsilon / l^2)}{l^2}, \\
\tilde{x} = \frac{x}{l}, \quad \tilde{t} = \frac{(l \mu / l)}{l^2}, \quad \tilde{\tau} = \frac{(l \mu / l)}{l^2}.
\end{align*}
\]

where for both cases \( \tilde{\tau} \) is the normalized charge relaxation time.

With either of these normalizations, Eqs. (1) reduce to

\[
\frac{\partial \tilde{E}}{\partial \tilde{t}} + \tilde{E} \frac{\partial \tilde{E}}{\partial \tilde{x}} = \tilde{J}.
\]

Integrating Eq. (4) between the electrodes yields

\[
\frac{d \tilde{E}^2}{d \tilde{t}} + \tilde{E} \frac{d \tilde{E}^2}{d \tilde{x}} + \frac{1}{2} \left[ \tilde{E}^2(1, \tilde{\tau}) - \tilde{E}^2(0, \tilde{\tau}) \right] = \tilde{J}.
\]

For a voltage source \( \tilde{U} = 1 \) and for a current source \( \tilde{J} = 1 \).

III. STEADY-STATE DISTRIBUTIONS

Setting the time derivative in Eq. (4) to zero and then


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where $\tilde{E}_0$ is the emitter electric field at $\tilde{\tau} = 0$ which must be specified as a boundary condition. For a voltage source excitation using the normalizations of Eqs. (2) the steady-state current $\bar{J}$ is obtained from Eq. (5) with $\tilde{\eta} = 1$. Figure 1(a) plots the steady-state distributions of electric field and space-charge density under space-charge-limited conditions ($\tilde{E}_0 = 0$) for various values of $\tilde{\tau}$. For high conductivity $\tilde{\tau}$ gets smaller and the electric field distribution approaches the uniform field of a resistor. Most of the space charge is near the emitting electrode. For all the plots the voltage is assumed constant, so the area under the electric field curves must be unity.

IV. TRANSIENT ANALYSIS

Equation (4) is a quasilinear partial differential equation which can be converted to a set of ordinary differential equations using the subsidiary equations obtained from the method of characteristics

\[
\frac{d\tilde{\tau}}{d\tilde{\tau}} = \frac{dx}{\tilde{E}} = \frac{d\tilde{E}}{\bar{J} - \tilde{E}/\tilde{\tau}}. \tag{7}
\]

The conservation of charge relation is recovered by taking the spatial derivative of Eq. (4)

\[
\frac{\partial \tilde{q}}{\partial \tilde{t}} + \frac{\tilde{q}}{\tilde{\tau}} + \tilde{E} \frac{\partial \tilde{q}}{\partial \tilde{x}} + \tilde{q}^2 = 0. \tag{8}
\]

Using the same reasoning as Eq. (7) the subsidiary equations are

\[
\frac{d\tilde{\tau}}{d\tilde{\tau}} = \frac{d\tilde{x}}{\tilde{E}} = \frac{d\tilde{q}}{\tilde{q}^2 - \tilde{q}/\tilde{\tau}}. \tag{9}
\]

Thus the equivalent set of ordinary differential equations are

\[
\frac{d\tilde{\tau}}{dt} = \tilde{E},
\]

\[
\frac{d\tilde{E}}{dt} = \bar{J} - \tilde{E}/\tilde{\tau},
\]

\[
\frac{d\tilde{q}}{dt} = \tilde{q} (\frac{1}{\tilde{\tau}}) - \tilde{q} = \frac{\tilde{q}_0}{(1 + \tilde{q}_0/\tilde{\tau}) \exp(\tilde{t} - \tilde{t}_0/\tilde{\tau}) - \tilde{q}_0/\tilde{\tau}}, \tag{10}
\]

where $\tilde{q}_0$ is the charge density at $\tilde{t} = \tilde{t}_0$ usually specified as an initial or boundary condition.

V. CURRENT EXCITATIONS

A. Step current from rest for space-charge-limited conditions

We now assume that at $t = 0$ the current is increased from zero to the value $\bar{J}$. For space-charge-limited conditions the emitter electric field is maintained at $E_0 = 0$ for all time. Using the normalizations of Eqs. (3), Eqs. (10) can be exactly integrated to

\[
\tilde{E}(x) + \tilde{E}_0\ln\left(\frac{\tilde{E} - \tilde{E}_0/\tilde{\tau}}{(\tilde{E} - \tilde{E}_0/\tilde{\tau})}\right) = \tilde{x}, \tag{6}
\]
FIG. 2. Trajectories $\tilde{x}(\tilde{t})$, distributions of electric field $\tilde{E}(\tilde{x})$, and space-charge density $\tilde{q}(\tilde{x})$ for $\tilde{v} = 1$ (solid lines) and $\tilde{v} = \infty$ (dotted lines). (a) Step current from rest and (b) open circuit from dc steady state.

$$\tilde{x} = \tilde{v}[\exp(-\tilde{t}/\tilde{t}_0) - 1] + \tilde{v}(\tilde{t} - \tilde{t}_0) + \tilde{x}_0,$$

$$\tilde{E} = \tilde{v}[1 - \exp(-\tilde{t}/\tilde{t}_0)],$$

$$\tilde{q} = 0 \quad \tilde{t}_0 = 0 \quad (11)$$

$$\tilde{E} = \frac{1}{\tilde{v}[\exp((\tilde{t} - \tilde{t}_0)/\tilde{v}) - 1]}, \quad \tilde{t}_0 > 0.$$  

Note that for the lossless case ($\tilde{v} = \infty$), Eqs. (11) reduce to

$$\tilde{x} = \frac{1}{2}(\tilde{t} - \tilde{t}_0)^2,$$

$$\tilde{E} = (\tilde{t} - \tilde{t}_0),$$

$$\tilde{q} = 0, \quad \tilde{t}_0 = 0,$$

$$\tilde{E} = \frac{1}{\tilde{v} - \tilde{t}_0}, \quad \tilde{t}_0 > 0. \quad (12)$$

Figure 2(a) plots the trajectories and distributions of the electric field and space-charge density of Eqs. (11) and (12) for $\tilde{v} = 1$ and $\tilde{v} = \infty$. Note the dark demarcation curve.
emanating from the origin of the charge trajectories which separates the effects of the initial conditions from the boundary conditions. Above the demarcation curve, the charge density is zero and the electric field is uniform. Below the demarcation curve the field and charge have their steady-state distributions, with only the spatial extent of the solution being a function of time. Once the demarcation curve reaches the other electrode ($\tau = 1$), the system is in the steady state. The solid curves in Fig. 1(b) show the terminal voltage versus time for various values of $\tau$.

B. Open circuit

After the system reaches the steady state, we suddenly open circuit the terminals. We continue to use the

![Graphs showing electric field and charge density distributions as functions of position and time.](image)

**Fig. 3.** Trajectories $\tilde{x}(\tau)$, distributions of electric field $\tilde{E}(\tilde{x})$, and space-charge density $\tilde{Q}(\tilde{x})$ for $\tilde{\tau} = 1$ (solid lines) and $\tilde{\tau} = \infty$ (dotted lines). (a) Step voltage from rest and (b) short circuit from dc steady state.
nondimensional quantities of Eqs. (3) using the charging current \( J \) as the normalizing current. The initial conditions are the steady-state solutions \( \bar{E}_{ss} \) and \( \bar{q}_{ss} \) for the step current. All trajectories will emanate only from the \( \bar{t} = 0 \) boundary as no further charge will be injected. The governing equations (10) are then
\[
\frac{d \bar{x}}{d \bar{t}} = \bar{x} - \bar{x}_0 = \bar{q}_{ss}(\bar{x}_0) \left( 1 - \frac{\bar{t}}{\bar{\tau}} \right) + \bar{x}_0,
\]
\[
\frac{d \bar{E}}{d \bar{t}} = -\frac{\bar{E}}{\bar{\tau}} - \bar{E} = \bar{E}_{ss}(\bar{x}_0) \exp \left( -\frac{\bar{t}}{\bar{\tau}} \right),
\]
\[
\bar{q} = \bar{q}_{ss}(\bar{x}_0) \left( 1 + \frac{\bar{q}_{ss}(\bar{x}_0)}{\bar{q}_{ss}(\bar{x}_0)} \exp(\bar{t}/\bar{\tau}) - \frac{\bar{q}_{ss}(\bar{x}_0)}{\bar{q}_{ss}(\bar{x}_0)} \right)^{-1},
\]
where \( \bar{x}_0 \) is the initial position of the charge trajectory at \( \bar{t} = 0 \). When \( \bar{\tau} = \infty \), Eqs. (13) reduce to
\[
\bar{x} = \bar{E}_{ss}(\bar{x}_0) \bar{t} + \bar{x}_0,
\]
\[
\bar{E} = \bar{E}_{ss}(\bar{x}_0),
\]
\[
\bar{q} = \bar{q}_{ss}(\bar{x}_0) \left[ 1 + \frac{\bar{q}_{ss}(\bar{x}_0)}{\bar{q}_{ss}(\bar{x}_0)} \right]^{-1},
\]
Figure 2(b) plots the open-circuit trajectories and distributions of field and charge for \( \bar{\tau} = 1 \) and \( \bar{\tau} = \infty \). Note that for those initial positions, such that
\[
\bar{q}_{ss}(\bar{x}_0) + \bar{x}_0 < 1,
\]
the initial charge is never completely swept out of the system. The dotted curves in Fig. 1(b) show the open-circuit voltage versus time for various values of \( \bar{\tau} \).

VI. VOLTAGE EXCITATIONS

A. Step voltage from rest for space-charge-limited conditions

We now consider the case when the voltage is instantaneously increased from zero to the value \( V \) at \( \bar{t} = 0 \). Using the normalizations of Eqs. (2), the governing equations (10) cannot be directly integrated as the current \( \bar{J} \) is not known but must be calculated from Eq. (5). However numerically integrating Eqs. (10) is easy as the equations are in exactly the right form for the Runge-Kutta method of numerical integration. The resulting charge trajectories and distributions of electric field and space-charge density are plotted in Fig. 3(a) for \( \bar{\tau} = 1 \) and \( \bar{\tau} = \infty \). Note the two dark demarcation curves emanating from the origin of the charge trajectories. This is due to spontaneous charge emission at \( \bar{t} = 0 \) to maintain the space-charge-limited condition at \( \bar{x} = 0 \). That is, along the \( \bar{t} = 0 \) boundary the field is uniform \( \bar{E} = 1 \) for all \( \bar{x} \). However at \( \bar{t} = 0, \) the field at \( \bar{x} = 0 \) must drop to 0. All the surface charge at \( \bar{t} = 0 \) instantaneously is emitted into the bulk as the surface charge drops to zero for all further time. Thus the left demarcation curve starts with unity slope, while the right demarcation curve starts with zero slope. The electric field at \( \bar{x} = 0, \bar{t} = 0 \) takes on all intermediate values. The charge density between the demarcation curves is only a function of time and not position
\[
\bar{q} = \frac{1}{\bar{\tau} \left( \exp(\bar{t}/\bar{\tau}) - 1 \right)}
\]