Charge injection between concentric cylindrical electrodes

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The transient electric field and space-charge distributions and terminal voltage-current characteristics are solved for drift-dominated unipolar ion conduction between concentric cylindrical electrodes. Special attention is directed to comparing solutions with charge injection from the inner and outer electrodes. As in past work with parallel-plate geometries, the method of characteristics is used to convert the governing partial differential equations into a set of first-order ordinary differential equations which are easily numerically integrated by the Runge-Kutta method. For current-source excitations the equations can be integrated exactly. Special cases examined include the charging transients to a step voltage or current excitation from rest under space-charge-limited conditions and the discharging transients for systems in the dc steady state which are instantaneously open or short circuited.

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I. INTRODUCTION

Past analyses of transient drift-dominated unipolar ion conduction consider a parallel-plate geometry so that regions with no charge have a uniform electric field. Reference 2 is a good source of references for past work related to conduction in solids. For identical electrodes, charge injection from either electrode results in the same solutions for the electric field and space-charge distributions and for the terminal voltage-current characteristics. This symmetry does not occur for concentric cylindrical electrodes since in the absence of space charge the electric field is nonuniform and is largest at the inner electrode. This paper continues previous work on charge injection between coaxial cylindrical electrodes by contrasting the solutions for charge injected from the inner and outer cylinders. In either case the electric field which propels the injected charge is due to both the self-field of the charge distribution and the $1/r$ field due to the imposed potential, present even if there is no space charge. Charges injected from the inner electrode find themselves in a stronger electric field than if injected from the outer cylinder.

We continue to use the method of characteristics to convert the governing partial differential equations into a set of ordinary differential equations which are valid for any initial and boundary conditions and for any type of applied excitation. For current-source excitations the equations can be exactly integrated, while for voltage-source excitations, the equations are in exactly the correct form for Runge-Kutta numerical integration. Special cases treated here include the charging transients to a step voltage or current source for space-charge-limited conditions at the injecting electrode and the discharging transients when the system is instantaneously open or short circuited from the dc steady state.

The results of this work showing the difference in behavior for charge injected from inner or outer cylinders may be useful in explaining polarity effects in breakdown studies in nonuniform field geometries such as between point-plane electrodes.

II. FIELD EQUATIONS

We consider the cylindrical geometry of depth $L$ in Fig. 1 where either the inner electrode at $r = R_i$ or the outer electrode at $r = R_o$ can be a source of ions with constant mobility $\mu$ in the dielectric medium. We assume the permittivity $\epsilon$ of the dielectric to be constant and that all field quantities can vary only with the radial coordinate $r$.

The governing field equations are then the irrotationality of the electric field $\mathbf{E}$, Gauss's law relating the electric field and space-charge density $q$, and conservation of current $J$ with a mobility conduction law ($\mathbf{J} = q\mu \mathbf{E}$),

$$\nabla \times \mathbf{E} = 0 - \int_{R_i}^{R_o} E_r dr = \nu, \quad (1)$$
$$\nabla \cdot (\epsilon \mathbf{E}) = q, \quad (2)$$
$$\nabla \cdot (q\mu \mathbf{E}) + \frac{\partial q}{\partial t} = 0, \quad (3)$$

where without loss of generality we assume in this work that the injected charge is positive so that the mobility $\mu$ in Eq. (3) is also positive.

When working with a voltage source $v$ it is convenient to normalize all variables to the outer radius $R_o$ and the steady-state voltage $V$ as

$$\tilde{r} = r/R_o, \quad \tilde{\nu} = \mu V/\epsilon R_o^2, \quad \tilde{v} = v/V, \quad \tilde{E} = E V, \quad \tilde{q} = q R_o^2/\epsilon V, \quad \tilde{t} = t R_o^2/(2\pi \epsilon \mu L V^2), \quad (4)$$

where $\tilde{t}$ is the total current flowing in the terminal wires and $E$ is the radial component of electric field.

![Fig. 1. That electrode with the positive potential is a source of positive ions with mobility $\mu$ in the insulating medium contained between coaxial cylindrical electrodes.](image-url)
Then the governing partial differential equation obtained from Eqs. (1)–(3) is
\[ \frac{\partial}{\partial \tilde{t}} (\tilde{E} \tilde{J}) + \frac{\partial}{\partial \tilde{r}} (\tilde{E} \tilde{J}) = - \tilde{I} \tilde{f}(\tilde{t}), \]  \( \tag{5} \)
where \( \tilde{I}(\tilde{t}) \), a function only of time and not position, is the total normalized current in the terminal wires due to displacement and conduction currents flowing in the dielectric. The upper (plus) sign in Eq. (5) is used for positive charge injection from the inner electrode, while the lower (negative) sign is used for positive injection from the outer electrode.

By dividing Eq. (5) by \( \tilde{r} \) and integrating over \( \tilde{r} \) from the inner to outer cylinders, the voltage is related to the current as
\[ \tilde{E} = \frac{1}{\ln \tilde{R}_i} \left( \frac{d \tilde{g}}{dt} + \frac{1}{2} [\tilde{E} \tilde{J}(\tilde{r}, t) - \tilde{E}(\tilde{R}_i, \tilde{t})] + \int_{\tilde{R}_i}^{1} \frac{\tilde{E} \tilde{J} \phi}{\tilde{r}} d\tilde{r} \right). \]  \( \tag{6} \)

Equations (5) and (6) are also appropriate for current-source excursions where we let \( \tilde{I} \) be the steady-state current and introduce the new normalizations
\[ \tilde{r} = \tilde{r} / \tilde{R}_i, \quad \tilde{t} = [\mu t / (2 \pi \varepsilon \varepsilon_0 \mu)]^{1/2}, \quad \tilde{E} = (2 \pi \varepsilon \varepsilon_0 \mu / \tilde{R}_i^{1/2}) E, \quad \tilde{q} = (2 \pi \varepsilon \varepsilon_0 \mu / \varepsilon_0 \mu a)^{1/2} q, \quad \tilde{v} = (2 \pi \varepsilon \varepsilon_0 \mu / \varepsilon_0 \mu a)^{1/2} v. \]  \( \tag{7} \)

### III. STEADY-STATE DISTRIBUTIONS

The dc steady-state distributions can be found by solving Eq. (5) with the time derivative set to zero and the current \( \tilde{I} \) independent of time. For charge injection from the inner electrode using the normalized variables of Eq. (4), the electric field distribution is
\[ \tilde{E}(\tilde{r}, \tilde{t} \rightarrow \infty) = (\tilde{J}^{1/2} / \tilde{r}) [\tilde{v}^2 - (1 - \tilde{E}_{\tilde{t}}^1 / \tilde{R}_i)]^{1/2}, \]  \( \tag{8} \)
where \( \tilde{E}_{\tilde{t}} \) is the normalized emitter electric field at the inner injecting electrode (\( \tilde{r} = \tilde{R}_i \)), which must be specified as a boundary condition and obeys the inequality
\[ 0 < \tilde{E}_{\tilde{t}} \leq 1 / (\ln \tilde{R}_i \ln \tilde{R}_o). \]  \( \tag{9} \)

For \( \tilde{E}_{\tilde{t}} \) negative the current would flow in the wrong direction for a passive medium, while for \( \tilde{E}_{\tilde{t}} > - (\ln \tilde{R}_i \ln \tilde{R}_o)^{-1} \), the divergence of \( E \) cannot be positive as required by Gauss’s law for positive charge injection and simultaneously satisfy Eq. (1) of having an average field strength between the electrodes proportional to the voltage. The total normalized current is found using Eq. (1) by integrating Eq. (8) between the electrodes:
\[ 1 = \tilde{J}^{1/2} \left[ (1 - a^2)^{1/2} - \left( \tilde{E}_{\tilde{t}} / \tilde{R}_i \right) - a \left( \cos^2 a - \sin^2 a / \tilde{R}_i \right) \right], \]  \( \tag{10} \)
where
\[ a^2 = \tilde{R}_i^2 (1 - \tilde{E}_{\tilde{t}}^1 / \tilde{R}_i), \quad b^2 = - a^2. \]  \( \tag{11} \)

In Eq. (10), both expressions are equivalent but it is more convenient to use the one which has real parameters \( a \) or \( b \). In general, the current cannot be written explicitly from Eq. (10) but must be solved numerically.

The steady-state results of Eqs. (8)–(11) are written for charge injection from the inner electrode at \( \tilde{r} = \tilde{R}_i \). If the voltage polarity is reversed so that positive charge is injected from the outer electrode at \( \tilde{r} = \tilde{R}_o \), the electric field is in the negative radial direction with the current \( \tilde{I} \) also flowing in the negative radial direction. If we use the negative sign in Eq. (5), the steady state electric field distribution is
\[ \tilde{E}(\tilde{r}, \tilde{t} \rightarrow \infty) = - (\tilde{I}^{1/2} / \tilde{r}) [1 - \tilde{r}^2 + \tilde{E}_{\tilde{t}}^2 / \tilde{I}^{1/2}], \]  \( \tag{12} \)
where \( - \tilde{E}_{\tilde{t}} \) is the normalized emitter electric field at the outer injecting electrode (\( \tilde{r} = \tilde{R}_o \)), which must be specified as a boundary condition and obeys the inequality
\[ 0 < \tilde{E}_{\tilde{t}} < 1 / (\ln \tilde{R}_i \ln \tilde{R}_o). \]  \( \tag{13} \)

The total normalized current is obtained by integrating Eq. (12) between the electrodes to obtain
\[ \tilde{J}^{1/2} \left[ (a^2 - 1)^{1/2} - (a^2 - \tilde{R}_o^2)^{1/2} \right], \]  \( \tag{14} \)
where
\[ a^2 = 1 + \tilde{E}_{\tilde{t}}^2 / \tilde{I}. \]  \( \tag{15} \)

The charge density for charge injection from either electrode is then given by
\[ \tilde{q} = [\tilde{I} / \tilde{r} \tilde{E}_{\tilde{t}}^1]. \]  \( \tag{16} \)

The steady-state results given thus far are valid for any charge-injection boundary condition. They reduce to simpler expressions for various conditions.

#### A. Space-charge-limited conditions

No matter which electrode is injecting charge, if an infinite amount of charge is available at the emitting electrode, the total emitter electric field must be zero to keep the current finite. This space-charge-limited boundary condition is often assumed in semiconductor analysis. Then if either \( \tilde{E}_{\tilde{t}} \) or \( \tilde{E}_{\tilo} \) is zero, the results reduce to the following:

- **charge injection from inner electrode**, \( \tilde{r} = \tilde{R}_i \)
  \[ \tilde{E}(\tilde{r}, \tilde{t} \rightarrow \infty) = (\tilde{J}^{1/2} / \tilde{r}) [\tilde{v}^2 - (1 - \tilde{E}_{\til{t}}^1 / \tilde{R}_i)]^{1/2}, \]
  \[ \tilde{q}(\tilde{r}, \tilde{t} \rightarrow \infty) = \tilde{J}^{1/2} [1 - \tilde{R}_i^2 a^2], \]
  \[ \tilde{I} = \left[ \ln \left( \frac{1 + (1 - \tilde{R}_i^2)^{1/2}}{\tilde{R}_i} \right) - (1 - \tilde{R}_i^2)^{1/2} \right] \]  \( \tag{17} \)
  \[ = [\cosh^2 a / (1 - \tilde{R}_i^2)]^{1/2} \]  \( \tag{18} \)

- **charge injection from outer electrode**, \( \tilde{r} = \tilde{R}_o \)
  \[ \tilde{E}(\tilde{r}, \tilde{t} \rightarrow \infty) = - (\tilde{J}^{1/2} / \tilde{r}) [1 - \tilde{r}^2 / \tilde{R}_o^2]^{1/2}, \]
  \[ \tilde{q}(\tilde{r}, \tilde{t} \rightarrow \infty) = \tilde{J}^{1/2} [1 - \tilde{R}_o^2 b^2], \]
  \[ \tilde{I} = (1 - \tilde{R}_o)^2. \]

#### B. Constant electric field distributions

Examining Eq. (8), we see that the electric field can be a constant for injection from the inner electrode if \( \tilde{E}_{\til{t}} = \tilde{I} \) so that
\[ \tilde{E}(\til{r}, \til{t} \rightarrow \infty) = \til{J}^{1/2} = \til{E}, \]
\[ \til{q}(\til{r}, \til{t} \rightarrow \infty) = \til{J}^{1/2} / \til{E}, \]
\[ \til{I} = (1 - \til{R}_i)^2. \]  \( \tag{18} \)
From Eq. (12) we see that there are no values of $\bar{E}_0$ and $\bar{I}$ which make $\bar{E}$ independent of radius for charge injection from the outer electrode.

C. Charge-free case ($\bar{q} = 0$)

If the steady-state charge density is to be zero between the electrodes, the current $\bar{I}$ must also be zero which requires that

$$\bar{E}_i = - (\bar{R}_1 \ln \bar{R}_2)^{-1}; \quad \bar{E}_o = - (\bar{R}_1 \ln \bar{R}_2)^{-1}. \tag{19}$$

For charge injection from either electrode the electric field distribution is then

$$\bar{E}(\bar{r}, \bar{t} \to \infty) = \mp (\bar{r} \ln \bar{R}_2)^{-1}. \tag{20}$$

IV. TRANSIENT ANALYSIS

Equation (5) falls into the class of quasilinear partial differential equations of first order which can be solved using the method of characteristics by writing the total differential of electric field as

$$d\bar{E} = \frac{\partial \bar{E}}{\partial \bar{r}} d\bar{r} + \frac{\partial \bar{E}}{\partial \bar{t}} d\bar{t}. \tag{21}$$

Equation (5) can be rewritten by expanding the derivatives as

$$\bar{E} \frac{\partial \bar{E}}{\partial \bar{r}} + \frac{\partial \bar{E}}{\partial \bar{t}} = \frac{\bar{i} - \bar{E}^2}{\bar{r}}. \tag{22}$$

By comparing Eq. (22) to Eq. (21), we see that along the trajectories

$$\frac{d\bar{r}}{d\bar{t}} = \frac{\bar{E}}{\bar{r}} \tag{23}$$

the electric field obeys the ordinary differential equation

$$\frac{d\bar{E}}{d\bar{t}} = \frac{\bar{i} - \bar{E}^2}{\bar{r}} - \frac{d}{d\bar{t}} (\bar{r} \bar{E}) = \frac{\bar{i}}{\bar{r}}. \tag{24}$$

The charge density along the trajectories of Eq. (23) can be similarly found by writing the total differential of volume charge as

$$d\bar{q} = \frac{\partial \bar{q}}{\partial \bar{r}} d\bar{r} + \frac{\partial \bar{q}}{\partial \bar{t}} d\bar{t}. \tag{25}$$

Using Eq. (2) in Eq. (3) the space charge obeys the equation

$$\bar{E} \frac{\partial \bar{q}}{\partial \bar{r}} + \frac{\partial \bar{q}}{\partial \bar{t}} + \bar{q}^2 = 0, \tag{26}$$

so that Eq. (25) can be rewritten as

$$\frac{d\bar{q}}{d\bar{t}} = -\bar{q}^2. \tag{27}$$

This equation can be directly integrated to yield the values of charge density along the characteristics of Eq. (23) as

$$\bar{q} = \frac{\bar{q}_0}{\left[1 + \bar{q}_0 (\bar{t} - \bar{t}_0)\right]^2}, \tag{28}$$

where $\bar{q}_0$ is the normalized charge density at time $\bar{t}_0$, to be determined from initial and boundary conditions.

If the current is known, it is easy to determine the electric field by direct integration of Eq. (24). If the voltage is known, the current must be found from Eq. (6). Since this relation also depends on the electric field distribution, numerical methods must be used.

For complete specification of the problem, it is necessary to supply the initial electric field or space-charge distribution as a function of position at $\bar{t} = 0$ and the electric field or charge density at the injecting electrode as a function of time.

In Secs. V-VIII case studies are examined for the charging and discharging transients for current and voltage excitations for positive charge injection from either electrode under space-charge-limited conditions. For most numerical work treated here, it is assumed that the normalized radius of the inner cylinder is $\bar{R}_1 = 0.25$. In all the plots, charge injection from the inner electrode at $\bar{r} = \bar{R}_1$ is drawn with dashed lines, while for charge injection from the outer electrode at $\bar{r} = 1$ the plots are drawn with solid lines.

V. STEP-CURRENT EXCITATION

For a step current, we put $\bar{i}(\bar{t}) = 1$ for $\bar{t} > 0$. For an initially unexcited system the initial conditions are

$$\bar{E}(\bar{r}, \bar{t} = 0) = \bar{q}(\bar{r}, \bar{t} = 0) = 0. \tag{29}$$

\begin{figure}[h]
\centering
\includegraphics[width=0.8\linewidth]{fig2}
\caption{Characteristic trajectories under space-charge-limited conditions for charge injection from the inner electrode (dashed curves) or from the outer electrode (solid curves) with $\bar{R}_1 = 0.25$. (a) Charging transient to a step current. (b) Discharging transients when system is instantaneously open-circuited from the dc steady state.}
\end{figure}
A. Charge injection from the inner electrode

For space-charge-limited conditions at \( \bar{\tau} = \bar{R}_i \) we have the boundary conditions

\[
E(\bar{\tau} = \bar{R}_i, \bar{q}) = 0, \quad \bar{q}(\bar{\tau} = \bar{R}_i, \bar{q}) = \infty.
\] (30)

In the \( \bar{\tau} \bar{q} \) plane, the solutions of Eqs. (23), (24), and (28) divide into regions as shown in Fig. 2(a) separated by the dark dashed demarcation curve emanating from the point \( \bar{\tau} = \bar{R}_i, \bar{q} = 0 \). Above the demarcation curve, all characteristics emanate from the \( \bar{\tau} = 0 \) boundary. Since the charge density is zero at \( \bar{\tau} = 0 \) we see that if we pick \( \bar{t}_0 = 0 \), then \( \bar{q}_0 = 0 \) so that the charge density is zero everywhere above the demarcation curve. Charge is injected at \( \bar{\tau} = \bar{R}_i \) where \( \bar{q}_0 = \infty \). The solutions to Eqs. (23), (24), and (28) are thus

\[
\bar{E} = \bar{\tau}/\bar{r}, \quad \bar{q} = 0
\]

on \( \bar{r}^2 = \bar{r}_0^2 + \bar{R}_i^2 \) (above demarcation curve)

\[
\bar{E} = (\bar{\tau} - \bar{t}_0)/\bar{r}, \quad \bar{q} = 1/(\bar{\tau} - \bar{t}_0)
\]

on \( \bar{r}^2 = (\bar{\tau} - \bar{t}_0)^2 + \bar{R}_i^2 \) (below demarcation curve),

where \( \bar{R}_i \) is the radius where those characteristics emanating from the \( \bar{\tau} = 0 \) boundary begin and \( \bar{t}_0 \) is the time those characteristics emanating from the \( \bar{\tau} = \bar{R}_i \) boundary begin. The demarcation curve is obtained by setting \( \bar{R}_i = \bar{R}_i \) and \( \bar{t}_0 = 0 \).

The demarcation curve reaches the outer electrode \( (\bar{\tau} = 1) \) at time \( \bar{t}_0 = (1 - \bar{R}_i^2)^{1/2} \). For \( \bar{\tau} > \bar{t}_0 \) the system has

FIG. 3. Electric field and space charge distributions for charge injection from space-charge-limited electrodes at \( \bar{\tau} = \bar{R}_i \) (dashed curves) or from \( \bar{\tau} = 1 \) (solid curves) with \( \bar{R}_i = 0.25 \). (a) Step current. (b) Open circuit from dc steady state.

FIG. 4. Time dependence of space-charge-limited charging and open-circuit voltages for \( \bar{R}_i = 0.25 \) with charge injection from either electrode.
its steady-state distributions given by Eq. (17) with $\tilde{T} = 1$ and using the normalizations of Eq. (7). For $0 < \xi < \tilde{T}_c$, steady-state distributions are below the demarcation curve, while the charge-free solution with the electric field having a $1/\xi$ dependence is above the demarcation curve. Integrating the electric field distribution over $\tilde{T}$ yields the voltage as

$$\tilde{v}(\tilde{T}) = \frac{1}{2} \left[ 1 - \ln(\tilde{T}^2 + \tilde{R}_i^2)^{1/2} - \tilde{R}_i \cos^{-1} \left( \tilde{R}_i / (\tilde{T}^2 + \tilde{R}_i^2)^{1/2} \right) \right],$$

$$0 < \tilde{T} < \tilde{T}_c$$

$$= (1 - \tilde{R}_i^2)^{1/2} - \tilde{R}_i \cos^{-1} \tilde{R}_i, \quad \tilde{T} > \tilde{T}_c.$$  \hspace{1cm} (32)

For the space-charge-limited case with $\tilde{R}_i = 0$, dashed curves in Fig. 2(a) show the trajectories, while the electric field and space-charge distributions are plotted in Fig. 3(a). The rising dotted curve in Fig. 4 shows the charging-voltage transient. Note in Fig. 3(a) how the charge density abruptly drops to zero when it crosses the demarcation curve. The electric field correspondingly then drops off as $1/\tilde{T}$ in the zero-charge region.

B. Charge injection from the outer electrode

With the same initial conditions of Eq. (29), the space-charge-limited boundary condition at $\tilde{T} = 1$ is now

$$\tilde{E}(\tilde{T} = 1, \tilde{T}) = 0, \quad \tilde{q}(\tilde{T} = 1, \tilde{T}) = 0.$$  \hspace{1cm} (33)

The analysis proceeds in the same way as for injection from the inner electrode, with the electric field now in the negative radial direction so that the charge trajectories of Eq. (23) have a negative slope as illustrated by the solid curves in Fig. 2(a). The solutions to Eqs. (23), (24), and (28) are now

$$\tilde{E} = -\tilde{T}/\tilde{r}, \quad \tilde{q} = 0$$

on $\tilde{T}^2 = -\tilde{T} + \tilde{R}_{sat}^2$ (below demarcation curve)

$$\tilde{E} = (\tilde{T} - \tilde{T}_c) / \tilde{r}, \quad \tilde{q} = 1 / (\tilde{T} - \tilde{T}_c)$$

on $\tilde{T}^2 = -\tilde{T} + 1$ (above demarcation curve),  \hspace{1cm} (34)

where again $\tilde{R}_{sat}$ is the radius where those characteristics emanating from the $\tilde{T} = 0$ boundary begin and $\tilde{T}_c$ is the time those characteristics emanating from the injecting $\tilde{T} = 1$ boundary begin. The demarcation curve is obtained by setting $\tilde{R}_{sat} = 1$ and $\tilde{T}_c = 0$. It reaches the other electrode at $\tilde{T} = \tilde{T}_c$ at the time $\tilde{t}_c = (1 - \tilde{R}_i^2)^{1/2}$ which is the same transport time for charge injection from the inner electrode. For times greater than $\tilde{T}_c$, the system has its steady-state distributions given by Eq. (17) with $\tilde{T} = 1$. The time dependence of the terminal voltage is

$$\tilde{v}(\tilde{T}) = \sqrt{\left( 1 - \ln \left( \frac{1 - \tilde{T}^2}{\tilde{R}_i} \right) \right) / \frac{1}{\tilde{T} - 1}}, \quad 0 < \tilde{T} < \tilde{T}_c$$

$$= (1 - \tilde{R}_i^2)^{1/2} - \ln \left( 1 + \frac{1}{\tilde{R}_i} \right), \quad \tilde{T} > \tilde{T}_c.$$  \hspace{1cm} (35)

The solutions for charge injection from the outer electrode can be compared in Figs. 2(a), 3(a), and 4 (solid curves) to those with charge injection from the inner electrode (dashed curves). Note in Fig. 3(a) that the magnitude of the electric field for charge injection from either electrode is the same in the region between the two demarcation curves before they intersect.

VI. OPEN CIRCUIT

We now assume that the space-charge-limited solutions for the step current have reached the steady state of Eq. (17) ($\tilde{T} = \tilde{T}_c$) when the system is instantaneously open circuited ($\tilde{T} = 0$). The electric field and space-charge distributions then slowly decay to zero from the initial conditions which are just the steady-state distributions of Eq. (17) due to a dc current. There is no further charge injection from either electrode.

From Eqs. (23), (24), and (28), the solutions for charge injection from the inner electrode are

$$\tilde{E}(\tilde{T}, \tilde{T}) = \left( \frac{1}{\tilde{R}_{sat}^2} - \tilde{R}_i^2 \right)^{1/2} \tilde{T}^{-1},$$

$$\tilde{q}(\tilde{T}, \tilde{T}) = \left[ \tilde{R}_i^2 - \tilde{R}_i \tilde{T}^2 \right]^{-1} \left[ \tilde{R}_i^2 - \tilde{R}_i \tilde{T}^2 \right]^{-1/2}.$$  \hspace{1cm} (36)

There is no demarcation curve since all characteristic trajectories emanate from the $\tilde{T} = 0$ boundary and none from the $\tilde{T} = \tilde{T}_c$ boundary where the electric field remains zero while the charge density decays as $\tilde{q}(\tilde{T} = \tilde{T}_c, \tilde{T}) = 1 / \tilde{T}$. The characteristic trajectories are shown in Fig. 2(b) (dashed curves), the field and charge distributions in Fig. 3(b) (dashed curves). The decaying voltage is given by

$$\tilde{v}(\tilde{T}) = \tilde{r} \ln \tilde{R}_i - 1 + (\tilde{T}^2 - \tilde{R}_i^2 + 1)^{1/2} - \tilde{R}_i \left( \tilde{T}^2 - \tilde{R}_i^2 \right)^{1/2}$$

$$\times \left( \frac{\cos^{-1} \left( \tilde{R}_i^2 / \tilde{T}^2 \right)^{1/2} - \cos^{-1} \left( \tilde{R}_i^2 / \tilde{T}^2 \right)^{1/2} }{\tilde{R}_i} \right),$$

$$0 < \tilde{T} < \tilde{T}_c$$

$$= \tilde{r} \ln \tilde{T}_c - 1 + \tilde{T}^2 - \tilde{R}_i \tilde{T}^2 \right)^{1/2}$$

$$\times \ln \left[ \frac{\tilde{R}_i \left( \tilde{T}^2 - \tilde{R}_i^2 \right)^{1/2} + (\tilde{T}^2 - \tilde{R}_i^2 + 1)^{1/2} }{\tilde{T} \left( \tilde{T}^2 - \tilde{R}_i^2 \right)^{1/2}} \right],$$

$$\tilde{T} > \tilde{T}_c$$  \hspace{1cm} (37)

and is plotted in Fig. 4 for $\tilde{R}_i = 0.25$ (dashed curve).

The results for charge injection from the outer electrode are

$$\tilde{E}(\tilde{T}, \tilde{T}) = -\left( 1 - \tilde{R}_{sat}^2 \right)^{1/2} / \tilde{T},$$

$$\tilde{q}(\tilde{T}, \tilde{T}) = \left( 1 - \tilde{T}^2 \right)^{1/2} \left[ 1 + (1 - \tilde{T}^2)^{-1} \right]^{-1}$$

with terminal voltage

$$\tilde{v}(\tilde{T}) = -\left( \tilde{T}^2 - 1 + \tilde{R}_i^2 \right)^{1/2} + \tilde{T} \ln \tilde{R}_i + 1$$

$$-\left[ \tilde{T}^2 + 1 \right]^{1/2} \ln \left( \frac{\tilde{T}^2 + 1}{\tilde{T}^2 + \tilde{R}_i^2} \right) \tilde{T}.$$  \hspace{1cm} (38)

These results can be compared to charge injection from the inner electrode in Figs. 2(b), 3(b), and 4.

VII. STEP-VOLTAGE EXCITATION

A step voltage $V$ is applied at $\tilde{T} = 0$ and remains constant thereafter. We continue to examine space-charge-limited conditions at the charge-injecting electrode, but now the initial conditions require a nonzero electric field distribution because its integral between the elec-
trodess must equal the voltage. Since space charge cannot instantaneously accumulate in the bulk, the initial conditions for injection from either electrode are

$$|E(r, t=0)| = |1/(4\pi \ln R)|, \quad \tilde{q}(r, t=0) = 0,$$  \hspace{1cm} (40)

where we now use the normalizations of Eq. (4). However, there is a discontinuity at the charge-injecting electrode since at $r=0$, the emitter electric field must instantaneously drop to zero to maintain the space-charge-limited condition. This discontinuity, also appearing in planar geometry, results in two demarcation curves as shown by the dark charging transient trajectories of Fig. 5(a). Between the two demarcation curves the electric field at $\tilde{t}=0$ takes on all intermediate values between zero and the initial value given by Eq. (40).

For those trajectories emanating from the $\tilde{t}=0$ boundary, the initial charge density is zero $\tilde{q}_b=0$ so that the charge density remains zero everywhere along those trajectories. Those trajectories emanating from the injecting electrode at time $t_0$ have infinite charge density $\tilde{q}_b=\infty$ so that the charge density decreases along the characteristics as $\tilde{q} = 1/(\tilde{t} - \tilde{t}_0)$. Between the two demarcation curves $\tilde{t}_0>0$ so that the charge density has a constant spatial distribution at a fixed time, decreasing with time as $\tilde{q} = 1/\tilde{t}$.

It is not possible to directly integrate Eqs. (23) and (24) because the current $I$ is not known and cannot be directly obtained from Eq. (6) until the spatial dependence of the electric field is known. However, Eqs. (23) and (24) are in exactly the correct form for the Runge-Kutta method of numerical integration, which combined with the trapezoidal rule for integration of the spatial integral in Eq. (6) allows easy numerical solutions. Note that the time derivative in Eq. (6) is zero for $\tilde{t} > 0$ and is a positive impulse at $\tilde{t} = 0$, representing the initial capacitive displacement current instantaneously charging the system.

Figures 5(a), 6(a), and 7 show the charging transient trajectories, the electric field and space-charge distributions, and the terminal current for charge injection from either electrode. We continue the policy of drawing as dashed lines the solution for charge injection from the inner electrode and as solid lines the solutions with injection from the outer electrode.

From Eq. (6), note that at $\tilde{t}=0$, the current steps up to the values

$$\tilde{I}(\tilde{t}=0) = -1/[2\tilde{R}_1^2(\ln \tilde{R}_1)^2], \text{ injection from } \tilde{r}=\tilde{R}_1$$

$$= 1/[2(\ln \tilde{R}_1)^2], \text{ injection from } \tilde{r}=1. \hspace{1cm} (41)$$

**TABLE 1.** Comparison of initial, steady-state, and short-circuit currents and the charge propagation times between electrodes with a step voltage for charge injection from the inner and outer cylinders for various values of inner radius $\tilde{R}_1$. Injection from the inner cylinder always results in larger currents and shorter propagation times.

<table>
<thead>
<tr>
<th>Step voltage $\tilde{I}(\tilde{t}=0)$</th>
<th>Short circuit $\tilde{I}(\tilde{t}=\infty)$</th>
<th>Charge injection from $\tilde{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{R}_1$ (Eq. (41))</td>
<td>$\tilde{R}_1$ (Eq. (42))</td>
<td>$\tilde{R}_1$</td>
</tr>
<tr>
<td>$\tilde{I}_1$</td>
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</tbody>
</table>

**FIG. 5.** Characteristic trajectories under space-charge-limited conditions for charge injection from either electrode with $\tilde{R}_1=0.05$. (a) Charging transient to a step voltage. (b) Discharging transient when system is instantaneously short-circuited from the dc steady state.
For small values of $R_i$ with injection from the inner electrode, the current initially decreases and then slightly rises to a small peak as it quickly approaches the steady-state values given in Eq. (17). For larger values of $R_i$, the current wave shape approaches that obtained with parallel-plate electrodes with a rise to a large pronounced peak and then some small oscillations as it approaches the steady state. With injection from the outer electrode, the current always increases to a peak.

Unlike the step-current case, we see in Fig. 5(a) that the time it takes the demarcation curves to reach the noninjecting electrode is longer when charge is injected from the outer electrode. At the time $t_{ed}$ defined as the time the first demarcation curve reaches the noninjecting electrode, the current reaches its peak value. The peak is sharper when charge is injected from the outer electrode, being especially noticeable for smaller inner radii. When the outer demarcation curve reaches the noninjecting electrode at time $t_i$, the system has almost reached its steady-state distributions.

Table 1 lists the initial and steady-state currents and the transient times $t_e$ and $t_i$ for charge injection from either cylinder for various inner-cylinder radii. The greatest differences are apparent for small values of $R_i$. The transient times for charge injection from the outer cylinder at $R=1$ are always longer than those for charge injection from the inner cylinder at $R=R_i$. We also note in Fig. 6(a) that the electric field at the inner cylinder for charge injection from the outer cylinder (solid lines) is always larger than the electric field at the outer cylinder for charge injection from the inner cylinder (dashed lines). Thus to reach a given maximum field strength a larger voltage is required for injection from the inner cylinder than for injection from the outer cylinder. For the same applied voltage the electric field
FIG. 7. Time dependence of space-charge-limited charging and short-circuit currents for various values of \(R_1\) for charge injection from either electrode. Note the time scale and amplitude differences between the charging and discharging cases.

 strengths and charge transport times can be significantly different depending on which electrode injects charge.

VIII. SHORT CIRCUIT

After the step voltage has been on for a long time, the electric field and space-charge distributions are given by Eq. (17). We now assume the system to be instantaneously short circuited (\(E = 0\)). We still use the normalized quantities of Eq. (4) normalized to the pre-short-circuit voltage.

Because the charge distribution remains continuous, the electric field instantaneously decreases by a term which has zero divergence so as not to change the charge density from Gauss’s law but to still satisfy Eq. (1) so that the average electric field between the cylinders is zero. This requires the electric field to be zero somewhere between the electrodes, decreasing in magnitude at the noninjeting electrode and reversing polarity at the injecting electrode.

To satisfy these constraints, the initial charge density is given by Eq. (17), while the electric field is decreased in magnitude from Eq. (17) by the term \(1/(\text{Fin} R_1)\). Since the emitter electric field reverses sign, no further charge can be injected. The calculation reduces to a pure initial-value problem with the solution depending only on the initial conditions since all characteristics emanate from the \(\tilde{t} = 0\) boundary.

Again, Eqs. (23) and (24) are solved by Runge-Kutta numerical integration together with the trapezoidal rule in Eq. (6).

The discharging transient trajectories are plotted in Fig. 5(b), while the field and charge distributions appear in Fig. 6(b). The short-circuit current is drawn in Fig. 7. The current reverses sign and instantaneously steps from the steady-state value of Eq. (17) \([I(\tilde{t} = 0\text{j})]\) to the new value

\[
\tilde{I}(\tilde{t} = 0\text{j}) = \tilde{I}(\tilde{t} = 0\text{j}) - \frac{[\tilde{I}(\tilde{t} = 0\text{j})]^{1/2}}{R_1/(\ln R_1)^{1/2}} \cos^{-1} \frac{R_1}{R_1^{1/2}}
\]

injection from \(\tilde{t} = \tilde{R}_1\)

\[
= I(\tilde{t} = 0) - \frac{[I(\tilde{t} = 0)]^{1/2}}{\ln R_1^{1/2}} \cosh^{-1} \frac{1}{R_1^{1/2}}
\]

injection from \(\tilde{t} = 1\).

IX. CONCLUDING REMARKS

The generalized transient solutions obtained here, summarized by Eqs. (23), (24), and (28), are true for drift-dominated unipolar conduction for one-dimensional radial charge transport between concentric cylindrical electrodes for any initial conditions, for any charge-injection boundary condition, and for any type of terminal constraint or excitation.

The examples treated here of the charging and discharging transients to voltage and current excitations were chosen because of their applicability to many experimental measurements and because of their analytical simplicity. More detailed computations, as have been performed for parallel-plate geometries, can examine the effects of excitation rise time, non-space-charge-limited boundary conditions, other excitations, and the addition of other charge carriers. The analysis presented in this paper may have applications in breakdown studies in other nonlinear field geometries since it has been shown that the charge transport time, the electric field and space-charge distributions, and the terminal current depend strongly on which electrode injects charge. In general, the charge transport time between electrodes is less and the terminal current is greater when the charge is injected in the high-field region (inner cylinder). The voltage necessary to reach breakdown field strengths at the non-charge-emitting electrode in the high-field region is also less for charge injection from the low-field electrode (outer cylinder). Thus, even though the charge transport time between electrodes is longer, the time delay to breakdown may be less because breakdown field strengths can be reached before the charge front reaches the opposite electrode. The discharge can then bridge the gap between the charge front and the electrode at an earlier time. Such discharges from the charge front to the opposite electrode have been observed for particulates traveling between electrodes and are termed microdischarges.

Recent work has shown the effects of polarity and gap length on the breakdown characteristics of mineral oil in a point-sphere geometry. It was found experimentally that for large gaps the breakdown voltage is
higher for a negative-point polarity, whereas for small gaps a negative point gives a lower breakdown voltage.

It is suggested that a coaxial geometry be considered for further breakdown measurements because the field nonuniformity is easily controlled by changing the radius of the inner cylinder, yet the analysis remains simple because the electric field is purely radial.

ACKNOWLEDGMENT

This work was supported by a National Science Foundation grant ENG 72-04214 A01.

\[6\] See any textbook on numerical integration, such as F.B. Hildebrand, Advanced Calculus for Applications (Prentice-Hall, Englewood Cliffs, N.J., 1965), pp. 102–106.
\[8\] M. Zahn and S.C. Pao, J. Electrostatics 1, 226 (1975).