FERROHYDRODYNAMIC TORQUE-DRIVEN FLOWS

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Ferrofluid motion driven by a traveling wave magnetic field can be in the same or opposite direction to the direction of wave propagation. A net time average body force on a ferrofluid is produced when there is a time phase lag between the ferrofluid magnetization $M$ and the driving magnetic field $H$. With $M$ and $H$ not collinear, there is also a dipole torque exerted on each magnetic domain which is balanced by the fluid viscous drag. Conservation of linear and angular momentum equations are solved in the steady state for a ferrofluid layer stressed by a traveling wave magnetic field. This configuration can be easily set up as an experiment, allows closed form solutions for the time average flow and spin fields, and exemplifies the essential physics of ferrohydrodynamic interactions.

1. Background

Although in recent years experiments have demonstrated ferrofluid pumping in traveling wave magnetic fields, the direction of fluid rotation has been reported in opposite directions by different investigators [1–5].

Following these experimental works, extensive analysis has been developed extending traditional viscous fluid flow to account for a nonsymmetric stress tensor that arises when $M$ and $H$ are not collinear and to then satisfy angular momentum conservation for internal spin fields (ref. [6], chap. 8). The purpose here is to take these now accepted equations for conservation of linear and angular momentum and apply them to a representative configuration that models typical rotating field experiments.

2. Governing magnetization equations

2.1. Magnetization

Under dc fields, the direction of magnetization $M$ is collinear with the applied field $H$. Then the body torque density $T = \mu_0 M \times H$, is zero. Under ac fields, the magnetization of each particle tries to follow the field but is opposed by viscosity or Brownian diffusion as the particle rotates to align its magnetic moment to the instantaneous field. Thus a phase lag develops between $M$ and $H$. The simplest description of this magnetization is then

$$\frac{dM}{dt} + \frac{1}{\tau} \left[ M - \frac{M_0 H}{H} \right] = 0,$$

where $M_0$ is the dc magnetization and $\tau$ may be the rotational viscous diffusion time $\tau_d = \rho R^2/15 \eta$ or Brownian rotational relaxation time $\tau_B = 4\pi \eta R^3/kT$ for particles of mass density $\rho$ and radius $R$ in a fluid with viscosity $\eta$ [7]. For typical ferrofluids with $\rho \approx 10$ g/cm$^3$, $R = 100$ Å and $\eta \approx 1$ cp, the time constants are $\tau_d \approx 10^{-10}$ s and $\tau_B \approx 10^{-6}$ s. Since $\tau_d \ll \tau_B$, effects due to particle inertia are essentially instantaneous while thermal fluctuations provide the significant phase shift between $M$ and $H$.

Assuming the sinusoidal steady state so that $M = \text{Re} \{ M e^{i\omega t} \}$ and neglecting the convective effects so that the convective time derivative in (1) becomes a partial time derivative, (1) yields the complex magnetic susceptibility, $\chi = \tilde{M}/\tilde{H}$, as

$$\tilde{M} = \chi_0 e^{-i\phi} \tilde{H} \Rightarrow \tilde{\chi} = \tilde{M}/\tilde{H} = \chi_0 e^{-i\phi},$$

where $\chi_0 = M_0/\{ H[1 + (\omega \tau)^2]^{1/2} \}$ is the magnitude of the magnetic susceptibility and $\phi =$...
\[ \tan^{-1}(\omega \tau) \] is the phase angle. The complex magnetic permeability is then
\[ \hat{\mu} = \mu_0 (1 + \hat{\chi}). \]

2.2. Magnetic field distribution

Consider the planar ferrofluid layer of thickness \( d \) in Fig. 1 excited by a traveling wave current sheet at \( x = 0 \)
\[ K_x = \text{Re}[\hat{K} \text{e}^{j(\omega t - kx)}]. \]

This configuration is the planar version of the stator winding of a cylindrical induction or synchronous machine where typically for an \( n \) pole machine three stator windings are wound each displaced by \( (240^\circ / n) \) in space and excited by currents with successive phase differences in time of \( 120^\circ \). These three standing wave excitations are equivalent to a rotating traveling wave of surface current. The ferrofluid is described by a complex magnetic permeability \( \mu \) given by (3). The current sheet is backed for \( x < 0 \) by an infinitely permeable \( (\mu \rightarrow \infty) \) magnetic core while the ferrofluid layer is surrounded from above by an infinite space of lossless material with real magnetic permeability \( \mu_2 \). Two typical limiting cases for \( \mu_e \) are for infinitely magnetically permeable iron (\( \mu_e \rightarrow \infty \)) or for free space (\( \mu_e = \mu_0 \)).

Because we assume that no free currents flow in any of the regions in Fig. 1 for \( x \neq 0 \), Ampere’s law requires the magnetic field \( \mathbf{H} \) to be curl free and thus representable as the gradient of a magnetic scalar potential \( \chi \)
\[ \nabla \times \mathbf{H} = 0 \rightarrow \mathbf{H} = \nabla \chi. \]

Then since each region has uniform magnetic permeability, Gauss’ law requires the magnetic scalar potential to obey Laplace’s equation
\[ \nabla \cdot \mathbf{H} = 0 \Rightarrow \nabla^2 \chi = 0. \]

Because of the form of the driving current in (4), the scalar potential in each region can be written in the similar form
\[ \chi = \text{Re}[\hat{\chi}(x) \text{e}^{j(\omega t - kx)}]. \]

Then the complex phasor amplitude \( \hat{\chi}(x) \) in each region satisfies Laplace’s equation of (6) as
\[ d^2 \hat{\chi}/dx^2 - k^2 \hat{\chi} = 0 \]
with the solution in each region in the form
\[ \hat{\chi}(x) = \begin{cases} 0 & x < 0, \\ \hat{\chi}_1 \text{e}^{-kx} + \hat{\chi}_2 \text{e}^{kx} & 0 < x \leq d, \\ \hat{\chi}_3 \text{e}^{k(x-d)} & x \geq d. \end{cases} \]

The magnetic field associated with these potentials is
\[ \mathbf{H} = \nabla \chi = \frac{d \hat{\chi}}{dx} \mathbf{i}_x - jk \hat{\chi} \mathbf{i}_z \]
\[ = \begin{cases} 0 & x < 0, \\ k \left\{ \begin{array}{l} -\hat{\chi}_1 \text{e}^{-kx} + \hat{\chi}_2 \text{e}^{kx} \\ -j[\hat{\chi}_1 \text{e}^{-kx} + \hat{\chi}_2 \text{e}^{kx}] \end{array} \right\} \mathbf{i}_x & 0 < x \leq d, \\ -k\hat{\chi}_3 \text{e}^{k(x-d)} \left\{ \mathbf{i}_x + j\mathbf{i}_z \right\} & x \geq d. \end{cases} \]

The amplitudes \( \hat{\chi}_1, \hat{\chi}_2 \) and \( \hat{\chi}_3 \) are found from the boundary conditions, that the tangential component of \( \mathbf{H} \) is discontinuous at \( x = 0 \) by the surface current and is continuous at \( x = d \) because there is no surface current there, and that the normal component of \( \mathbf{B} = \mu \mathbf{H} \) field is continuous at \( x = d \)
\[ \hat{H}_z(x = 0_+) = -K_z \Rightarrow \mu \hat{H}_z(x = d) = \hat{K}_z, \]
\[ \hat{H}_z(x = d_+) = \hat{H}_z(x = d) \Rightarrow \hat{\chi}_1 \text{e}^{-kd} + \hat{\chi}_2 \text{e}^{kd} = \hat{\chi}_3, \]
\[ \mu_e \hat{H}_z(x = d_+) = \hat{\mu} \hat{H}_z(x = d) \Rightarrow \mu \hat{\chi}_3 \]
\[ = -\hat{\mu} \left[ \hat{\chi}_1 \text{e}^{-kd} + \hat{\chi}_2 \text{e}^{kd} \right]. \]
Solutions to (11) are
\[ \hat{x}_1 = \frac{-\hat{X}_2 e^{2kd}[1 + \hat{\mu}/\mu_e]}{[1 - \hat{\mu}/\mu_e]} = \frac{\hat{X}_3 e^{kd}}{(2\hat{\mu}/\mu_e)} \left( 1 + \frac{\hat{\mu}}{\mu_e} \right) \]
\[ = \frac{j\hat{\kappa}_y [1 + \hat{\mu}/\mu_e] e^{2kd}}{k [1 - \hat{\mu}/\mu_e - e^{2kd}(1 + \hat{\mu}/\mu_e)]}. \]

**2.3. Time average magnetic force density distribution**

The magnetic force density on the ferrofluid layer in the region \(0 < x < d\) is
\[ f = \mu_0 (M \cdot \nabla) H. \]

The traveling wave field and magnetization components in the form of (7) are
\[ H = \text{Re} \left[ \left( \hat{H}_x(x) i_x + \hat{H}_z(x) i_z \right) e^{i(\omega t - kx)} \right], \]
\[ M = \text{Re} \left[ \chi_0 e^{-\gamma \mu} \left[ \hat{H}_x(x) i_x + \hat{H}_z(x) i_z \right] \times e^{i(\omega t - kx)} \right], \]

so that the time average force density components are computed as (ref. [8], section 2.15)
\[ \langle f \rangle = \frac{i}{2} \mu_0 \text{Re} \left( \chi_0 e^{-\gamma \mu} \left[ \hat{H}_x \frac{d}{dx} + jk\hat{H}_z \right] \right) \]
\[ \times \left[ \hat{H}_x^* i_x + \hat{H}_z^* i_z \right] \]
\[ = \frac{i}{2} \mu_0 \text{Re} \left( \chi_0 e^{-\gamma \mu} \left[ i_x \left( \hat{H}_x \frac{d}{dx} + jk\hat{H}_z \right) \right. \right] \]
\[ + i_x \left( \hat{H}_z \frac{d}{dx} + jk\hat{H}_x \right) i_z \left. \right), \]

where \( \hat{H}_x^* \), \( \hat{H}_z^* \) are the complex conjugate amplitudes of the magnetic field components. Substituting the field components of (10) for \(0 < x \leq d\) into (15) gives the time average force density components
\[ \langle f_x \rangle = -\mu_0 \chi_0 k^3 \left[ |\hat{x}_1|^2 e^{-2kx} + |\hat{x}_2|^2 e^{2kx} \right] \]
\[ \times \cos \phi, \]
\[ \langle f_z \rangle = \mu_0 \chi_0 k^3 \left[ |\hat{x}_1|^2 e^{-2kx} + |\hat{x}_2|^2 e^{2kx} \right] \sin \phi. \]

**2.4. Time average magnetic torque density distribution**

The magnetic torque density on the ferrofluid layer in the region \(0 < x < d\) is
\[ T = \mu_0 M \times H = \mu_0 \left[ -M_x H_z + M_z H_x \right] i_y. \]

The time average torque density is
\[ \langle T_x \rangle = \frac{i}{2} \mu_0 \text{Re} \left( \chi_0 e^{-\gamma \mu} \left[ -\hat{H}_x \hat{H}_z^* + \hat{H}_z \hat{H}_x^* \right] \right) \]
\[ = \mu_0 k^2 \chi_0 \left[ |\hat{x}_1|^2 e^{-2kx} - |\hat{x}_2|^2 e^{2kx} \right] \sin \phi. \]

**3. Governing fluid mechanical equations**

**3.1. Conservation of linear and angular momentum**

We consider an incompressible Newtonian fluid of mass density \(\rho\) and dynamic viscosity \(\eta\) moving with linear velocity \(v\). However, this fluid also has an internal spin field \(s = I \omega\) where \(I\) is the average fluid moment of inertia density and \(\omega\) is the angular spin rate (ref. [6], chap. 8). The spin field is due to the rotation of magnetic colloid particles together with viscous fluid that is locally entrained by the spinning particles and contributes to the total angular momentum of the ferrofluid. Angular momentum is coupled to the linear momentum through an antisymmetric stress tensor proportional to the difference between half the vorticity, \(\nabla \times v\), and angular spin rate \(\omega\). The proportionality coefficient is termed the vortex viscosity \(\zeta\). For incompressible fluids such that
\[ \nabla \cdot v = 0, \quad \nabla \cdot \omega = 0, \]
the coupled linear and angular momentum conservation equations for force density \(f\) and torque density \(T\) for a fluid in a gravity field \(-g_i\) are
\[ \rho \frac{Dv}{Dt} = -\nabla p + f + 2\zeta \nabla \times \omega + (\zeta + \eta) \nabla^2 v - \rho g_i, \]
\[ I \frac{D\omega}{Dt} = T + 2\zeta (\nabla \times v - 2\omega) + \eta \nabla^2 \omega. \]
where \( p \) is the pressure and \( \eta' \) the bulk coefficient of the spin viscosity.

### 3.2. Viscous dominated flow and spin

We assume that the planar ferrofluid layer of fig. 1 has viscous dominated flow so that inertia is negligible, is in the steady state, and responds to the time average magnetic force and torque densities. Then the flow and spin velocities are of the form

\[
v = v_x(x)i_x, \quad \omega = \omega_x(x)i_x
\]

and (20) and (21) reduce to

\[
0 = - \nabla p + \langle f \rangle + \left[ 2\xi \frac{\partial \omega_x}{\partial x} + (\xi + \eta) \frac{\partial^2 v_x}{\partial x^2} \right] i_x - \rho g i_x.
\]

(22)

(23)

From (16) we know that the magnetic force density has \( x \) and \( z \) components that only depend on \( x \). Following the method of Melcher (ref. [8], section 9.3) we define a pseudo-energy function \( \xi \), such that

\[
\langle f_x \rangle = -\frac{d\xi}{dx} \Rightarrow \xi = \frac{1}{2} \mu_0 \chi_0 k^2 \left[ |\hat{x}_1|^2 e^{-2kx} + |\hat{x}_2|^2 e^{2kx} \right]
\]

(24)

\[
\times \cos \phi
\]

(25)

and then define a modified pressure that also includes a gravitational head term

\[
p' = p + \xi + \rho g x.
\]

(26)

so that (23) can be expressed as

\[
0 = - \nabla (p') + \left[ \langle f_x \rangle + 2\xi \frac{\partial \omega_x}{\partial x} + (\xi + \eta) \frac{\partial^2 v_x}{\partial x^2} \right] i_x.
\]

(27)

This use of \( p' \) in (27) is allowed because \( \xi \) and the gravity term are only a function of \( x \) and not \( z \). Then the \( x \) component of (27) shows that \( p' \) is not a function of \( x \) and that \( \partial p'/\partial z \) must be a constant as the bracketed term is only a function of \( x \) and not \( z \). Then

\[
2\xi \frac{d\omega_x}{dx} + (\xi + \eta) \frac{d^2 v_x}{dx^2} = \frac{\partial p'}{\partial z} - \langle f_z \rangle
\]

\[
= \frac{\partial p'}{\partial z} - \mu_0 k^3 \chi_0 \left[ |\hat{x}_1|^2 e^{-2kx} + |\hat{x}_2|^2 e^{2kx} \right]
\]

\[
\times \sin \phi.
\]

(28)

\[
2\xi \left( \frac{d^2 v_x}{dx^2} + 2\omega_x \right) + \eta' \frac{d^2 \omega_x}{dx^2} = \langle T_x \rangle
\]

\[
= \mu_0 k^2 \chi_0 \left[ |\hat{x}_1|^2 e^{-2kx} - |\hat{x}_2|^2 e^{2kx} \right] \sin \phi.
\]

(29)

Eliminating \( v_x \) terms in (28) and (29) gives a single differential equation in the spin velocity

\[
\eta' \frac{d^3 \omega_x}{dx^3} - \frac{4\xi \eta}{(\xi + \eta)} \frac{d\omega_x}{dx} = \frac{2\xi}{(\xi + \eta)} \left( \frac{\partial p'}{\partial z} - \langle f_z \rangle \right) - \frac{d}{dx} \langle T_x \rangle
\]

\[
= \frac{2\xi}{(\xi + \eta)} \frac{\partial p'}{\partial z} + \frac{2\eta}{(\xi + \eta)} \mu_0 \chi_0 k^3
\]

\[
\times \sin \phi \left[ |\hat{x}_1|^2 e^{-2kx} - |\hat{x}_2|^2 e^{2kx} \right].
\]

(30)

### 4. Solutions

#### 4.1. Linear and spin velocities

The solutions to (30) are found from the superposition of particular solutions to each of the driving terms on the right of (30), and homogeneous solutions when the driving terms are all zero. The spin velocity is then

\[
\omega_x(x) = W_0 - \frac{1}{2\eta} \frac{\partial p'}{\partial z} x + W_1 e^{\gamma x} + W_2 e^{-\gamma x}
\]

\[
W_1 e^{-2\gamma x} + W_2 e^{2\gamma x}.
\]

(31)
where

\[ W_{p1.2} = \frac{\eta}{\eta' (s + \eta)} \mu_0 x_0 k^2 \sin \phi |\hat{X}_{1.2}|^2, \quad (32) \]

\[ r_0 = \left[ \frac{4 s \eta}{\eta' (s + \eta)} \right]^{1/2}. \]

The linear velocity \( v_2(x) \) is most easily found by solving (29) for the derivative of \( v_2 \) and then integrating to obtain

\[ v_2(x) = V_0 - 2 W_0 x + \frac{1}{2 \eta} \frac{\partial p'}{\partial z} x^2 \]

\[ - s \frac{2 s}{(s + \eta) r_0} \left[ W_1 e^{\omega x} - W_2 e^{-\omega x} \right] \]

\[ + s \frac{k \eta'}{\eta} \left[ W_{p1} e^{-2 \omega x} + W_{p2} e^{2 \omega x} \right]. \quad (33) \]

Boundary conditions at the planar flow channel walls are

\[ v_2(x = 0) = 0, \quad \omega_2(x = 0) = 0, \]

\[ v_2(x = d) = 0, \quad \omega_2(x = d) = 0, \quad (34) \]

so that the constants \( W_0, W_1, W_2 \) and \( V_0 \) are

\[ W_0 = 2 \left[ (B - D) \sinh r_0 d \right. \]

\[ + \beta (A + C) (1 - \cosh r_0 d) \left] / \text{Det}, \right. \]

\[ W_1 = \left\{ 2 A d \left[ (\beta / 2 d) (1 - e^{-\omega d}) - e^{-\omega d} \right] \right. \]

\[ - (B - D) (1 - e^{-\omega d}) \]

\[ + 2 C d \left[ (\beta / 2 d) (e^{-\omega d} - 1) + 1 \right] / \text{Det}, \right. \]

\[ W_2 = \left\{ 2 A d \left[ (\beta / 2 d) (1 - e^{\omega d}) + e^{\omega d} \right] \right. \]

\[ + (B - D) (1 - e^{\omega d}) \]

\[ + 2 C d \left[ (\beta / 2 d) (e^{\omega d} - 1) - 1 \right] \left] / \text{Det}, \right. \]

\[ V_0 = \left\{ 4 \beta A d [(\beta / 2 d) \sinh r_0 d - \cosh r_0 d] \right. \]

\[ + 4 B d [(\beta / 2 d) (1 - \cosh r_0 d) + \sinh r_0 d] \right. \]

\[ + 4 \beta C d \left[ (\beta / 2 d) \sinh r_0 d + 1 \right] \right. \]

\[ + 2 \beta D [1 - \cosh r_0 d] \left] / \text{Det}, \right. \]

\[ \text{Det} = 4 d \left[ \sinh r_0 d + (\beta / d) (1 - \cosh r_0 d) \right]. \quad (35) \]

where

\[ \beta = 2 s / (s + \eta) r_0, \]

\[ A = - W_{p1} + W_{p2}, \]

\[ B = - k \eta' / \eta (W_{p1} + W_{p2}), \]

\[ C = d / 2 \eta \frac{\partial p'}{\partial z} - W_{p1} e^{-2 k d} + W_{p2} e^{2 k d}, \]

\[ D = - d^2 / 2 \eta \frac{\partial p'}{\partial z} - k \eta' / \eta [W_{p1} e^{-2 k d} + W_{p2} e^{2 k d}]. \quad (36) \]

4.2. Flow rate and pressure rise

The flow rate through the channel is

\[ Q = \int_0^d v_2 \, dx \]

\[ = V_0 d - W_0 d^2 + \frac{d^3}{6 \eta} \frac{\partial p'}{\partial z} \]

\[ - \frac{\beta}{r_0} \left[ W_1 (e^{\omega d} - 1) + W_2 (e^{-\omega d} - 1) \right] \]

\[ - \frac{\eta'}{2 \eta} \left[ W_{p1} (e^{-2 k d} - 1) - W_{p2} (e^{2 k d} - 1) \right]. \quad (37) \]

Fig. 2 plots the magnetic field driven flow rate \( Q \) (dotted lines) when the pressure gradient, \( \partial p' / \partial z \), is zero and the pressure gradient (solid lines) when the flow rate \( Q \) is zero. For \( \partial p' / \partial z \)
Fig. 3. The non-dimensional (a) linear velocity distribution $c_x = c_x(d)/(p,\omega_o, (\kappa d)^2$ and (b) spin velocity distribution, $\phi_s = \phi_s/(\omega_o, (\kappa d)^2$ for the same parameters given in fig. 2 and $kd = 1$. The solid lines have zero pressure rise while the dotted lines have zero net flow.

positive, the fluid will be displaced in the direction of the traveling wave. This pressure rise could support a static head of fluid. Fig. 3 shows the corresponding linear and spin velocity distributions.

References