Space charge dynamics of viscous liquids

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A general set of relations, useful in the analysis of the propagation and instability characteristics of small signal space charge and polarization interfacial waves, are derived for field and flow variables on the perturbed surfaces of a uniformly charged, incompressible, viscous prototype layer.

Recent work has studied the stability properties of small signal electro-fluid mechanical perturbations in nonhomogeneous fluids which are incompressible and perfectly insulating with stratifications of mass density $\rho$, space charge density $q$, electric field intensity $E$, and permittivity $\varepsilon$. In general for this electrohydrodynamic case, incipience of instability occurs independent of viscosity at zero frequency. A necessary and sufficient condition for stability of stratifications in the $x$ direction with $g$ in the negative $x$ direction and $E$ defined in the positive $x$ direction is \[ \frac{d}{dx} \int_D \frac{d}{dx} = 0. \] (1)

If only conditions at incipience of instability are needed, the eigenvalue problem is solved with time rates set to zero so that the eigenvalue problem in frequency $\omega$ is traded for an eigenvalue problem in the system parameters, such as the electric field. However, if wave propagation characteristics or growth rates of an instability are needed, the whole set of perturbation differential equations must be solved, subject to the boundary conditions. In general, this is difficult, because the coefficients of the differential equations are functions of position, although recent work has solved these equations for the special cases of weak gradient and exponential stratifications for which the differential equations reduce to the constant coefficient type.

The analysis has also been developed for interfacial waves in systems composed of superposed layers where physical properties are constant within a layer, so that the governing differential equations have constant coefficients. A systematic approach for handling multilayered systems is developed through the use of general prototype layers, which are described with sufficient generality so as to relate the perturbation interfacial field and flow variables of pressure, displacement, electric potential, and electric displacement. Since these variables appear in the interfacial boundary conditions, many layers can be treated by simply “splicing” regions together in a manner dictated by the boundary conditions.

A limitation of this work is that the fluids of interest are assumed inviscid. Since the stability criterion of (1) is independent of viscosity, at first glance it appears that viscosity only adds the complication of damping terms for propagating waves or limits the growth rate of an instability. However, it is important to realize that viscosity must be included in those cases where perturbation electrical shear stresses occur, as a model cannot be consistently formulated without a physical mechanism for balancing these forces. Two examples for which perturbation electrical shear stresses exist are for the cases of either tangential or perpendicular electric fields at an interface between two liquids with different electrical conductivities. However, if the interface is perfectly conducting, there are no electrical shear stresses since then the electric field terminates normal to the interface resulting in no electrical shear force.

In this paper, we wish to extend earlier electrohydrodynamic analysis by including fluid viscosity in the derivation of general relations for the viscous prototype layer shown in Fig. 1. With the inclusion of viscous forces, the general relations must now include shear as well as normal stresses at each interface. However, none of the electrical terms will be affected by the viscosity.

The development proceeds using the Navier–Stokes equation of conservation of momentum including electrical, gravity, and viscous forces, conservation of mass for an incompressible liquid, and Maxwell’s equations of the irrotationality of the electric field, Gauss’ law, and charge

![FIG. 1. Planar prototype layer for a perfectly insulating, incompressible, and viscous fluid supporting a uniformly distributed charge density.](image)
conservation for a perfectly insulating incompressible liquid. In the prototype layer, the fluid is homogeneous such that the properties of mass density \( \rho \), charge density \( q \), dielectric constant \( \varepsilon \), and viscosity \( \mu \) remain constant even for small signal motions. Since the equilibrium is stationary, viscosity has no effect. As the perturbation equations are linear, constant coefficient, solutions for the perturbation variables can be assumed of the form

\[
p' = \text{Re} \tilde{p}(x) \exp[j(\omega t - k x)],
\]

where because of the isotropy of the system, all variations with the coordinate \( y \) are neglected. The perturbation pressure \( p' \) and perturbation electric potential \( \phi' \) must obey Laplace's equation and further defining the modified pressure term

\[
\sigma' = p' + q_k \phi',
\]

the perturbation equations can be summarized as

\[
\begin{align*}
 j \omega p u_x + D \sigma &= \mu_k (D^2 - k^2) u_x, \\
 j \omega p u_y - jk \sigma &= \mu_k (D^2 - k^2) u_y, \\
 D u_x - jk u_y &= 0, \\
 (D^2 - k^2) \left( \frac{\partial}{\partial \rho} \right) &= 0.
\end{align*}
\]

The solutions to (3)-(6) are

\[
\begin{align*}
\tilde{u}_x(x) &= A \sinh k x + B \cosh k x + C \sinh \delta x + D \cosh \delta x, \\
\tilde{u}_x(x) &= -j[A \cosh k x + B \sinh k x + (\delta/k)C \cosh \delta x + (\delta/k)D \sinh \delta x], \\
\tilde{u}_y(x) &= (-j \omega p u_0 / k)(A \cosh k x + B \sinh k x), \\
\tilde{\phi}(x) &= E \sinh k x + F \cosh k x,
\end{align*}
\]

where

\[
\delta = [k^2 + (j \omega p / \mu_k)]^{1/2}.
\]

The interfacial velocities are related to the interfacial displacements \( \xi \) as

\[
\tilde{v}_x^{\alpha \beta} = j \omega \tilde{\xi}_x^{\alpha \beta}.
\]

The relations between the electric fields normal to the interface to the perturbation potentials and velocities are the same whether the fluid is viscous or inviscid. From (10) or from the earlier inviscid analysis these relations are

\[
\begin{bmatrix}
\tilde{e}_x \tilde{e}_x^{\alpha \beta} \\
\tilde{e}_x \tilde{e}_y^{\alpha \beta}
\end{bmatrix} = \begin{bmatrix}
-\tilde{e}_x k \cosh k \Delta & \tilde{e}_x k \sinh k \Delta \\
-\tilde{e}_y k \sinh k \Delta & \tilde{e}_x k \cosh k \Delta
\end{bmatrix} \begin{bmatrix}
\tilde{\phi}^{\alpha \beta} + E^{\alpha \beta} \tilde{\xi}^{\alpha \beta} \\
\tilde{\xi}^{\alpha \beta}
\end{bmatrix} + q_k \begin{bmatrix}
\tilde{\xi}^x \\
\tilde{\xi}^y
\end{bmatrix}.
\]

It is important to realize that (12) includes the total first-order change in all variables evaluated at the interfaces, taking into account that as the interfaces deform, in addition to perturbing all variables, the equilibrium quantities acting on the interfaces also change.

For the mechanical relations, we need to relate the constants \( A, B, C, \) and \( D \) to the interfacial velocities, where without loss of generality we set \( \beta = 0 \) and \( \alpha = \delta \).

\[
\begin{align*}
\tilde{v}_x^{\alpha} &= \tilde{v}_x(\Delta); \\
\tilde{v}_y^{\alpha} &= \tilde{v}_y(\Delta), \\
\tilde{v}_x^{\delta} &= \tilde{v}_x(0); \\
\tilde{v}_y^{\delta} &= \tilde{v}_y(0).
\end{align*}
\]

Then, using the definitions of (13) in (7) and (8) we obtain the relations

\[
\begin{bmatrix}
\tilde{v}_x^m \\
\tilde{v}_y^m
\end{bmatrix} = \begin{bmatrix}
\sinh k \Delta & \cosh k \Delta \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\sinh \delta \Delta & \cosh \delta \Delta \\
-j \frac{k}{\delta} \cosh \delta \Delta & -j \frac{k}{\delta} \sinh \delta \Delta
\end{bmatrix} \begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}.
\]

In our method it is necessary to invert this relation and then to relate interfacial stresses to the interfacial velocities. The interfacial stresses include those due to viscosity and the hydrodynamic pressure. The total hydrodynamic stress is of the form

\[
\sigma_{xy} = \mu_k \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) - p \delta_y,
\]

so we obtain

\[
\hat{S}_{xx}(\alpha) = 2 \mu_k D \delta v_{x \alpha} - \rho(\alpha) = \hat{S}_x(\alpha),
\]

\[
\hat{S}_{xx}(\beta) = 2 \mu_k D \delta v_{x \beta} - \rho(\beta) = \hat{S}_x(\beta),
\]

where

\[
\hat{S}_{xx}(\alpha) = \mu_k \left[ D \delta v_{x \alpha} - j k \delta v_{y \alpha} \right] = \hat{S}_x(\alpha),
\]

\[
\hat{S}_{xx}(\beta) = \mu_k \left[ D \delta v_{x \beta} - j k \delta v_{y \beta} \right] = \hat{S}_x(\beta).
\]

Then, taking into account first-order changes due to changes in the equilibrium quantities acting on the interfaces as they move, (16) yields the general relations between shear and normal stresses to the interfacial velocities and potentials as

\[
\begin{bmatrix}
\hat{S}_x^{\alpha} \\
\hat{S}_y^{\alpha} \\
\hat{S}_x^{\delta} \\
\hat{S}_y^{\delta}
\end{bmatrix} = \begin{bmatrix}
(a + h) & b & c & d \\
-b & -(a + h) & d & c \\
-d & -c & e & f \\
d & -c & -f & -e
\end{bmatrix} \begin{bmatrix}
\tilde{v}_x^m \\
\tilde{v}_y^m \\
\tilde{\xi}_x \\
\tilde{\xi}_y
\end{bmatrix} + q_k \begin{bmatrix}
\tilde{\xi}_x^m \\
\tilde{\xi}_y^m
\end{bmatrix}.
\]
where

\[ a = -\mu_0 \left( \frac{\delta^2 - k^2}{kD} \right) \left( \frac{k}{\delta} \coth \delta \Delta - \coth k \Delta \right), \]

\[ b = \mu_0 \left( \frac{\delta^2 - k^2}{kD} \right) \left( \frac{k}{\delta} \csch \delta \Delta - \csch k \Delta \right), \]

\[ c = \frac{j\mu_0}{D} \left[ \delta \left( 1 + \frac{3k^2}{\delta^2} \right) \right. \]
\[ \times (\csch k \Delta \csch \delta \Delta - \coth k \Delta \coth \delta \Delta) \]
\[ \left. + k \left( 3 + \frac{k^2}{\delta^2} \right) \right], \]

\[ d = -\frac{j\mu_0 (\delta^2 - k^2)}{\delta D} (\coth \delta \Delta \csch k \Delta \]
\[ - \coth k \Delta \csch \delta \Delta), \]

\[ e = -\frac{\mu_0 (\delta^2 - k^2)}{\delta D} \left( \frac{k}{\delta} \coth k \Delta - \coth \delta \Delta \right), \]

\[ f = -\frac{\mu_0 (\delta^2 - k^2)}{\delta D} (\csch \delta \Delta - \frac{k}{\delta} \csch k \Delta). \]

\[ h = \rho \sigma g / j \omega, \]

\[ D = 1 + \frac{k^2}{\delta^2} + \frac{2k}{\delta} (\csch k \Delta \csch \delta \Delta - \coth k \Delta \coth \delta \Delta). \]

Equations (12) and (17) provide the complete description of a uniformly charged viscous layer of incompressible liquid. They can help describe a great variety of physical interactions of multilayered systems including the effects of gravity, polarization, space charge, viscosity, and the boundaries. Similar relations have been derived for applications to electrochemical phenomena but did not include bulk Coulomb forces.\(^7\)

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