Finite Element Based Kerr Electro-Optic Reconstruction of Space Charge

A. Ústündag\textsuperscript{1} and M. Zahn
M.I.T.
Dept. of Elect. Eng. and Comp. Sciences
Cambridge, MA

ABSTRACT

Recently we used the onion peeling method to reconstruct the axisymmetric electric field distribution of point/plane electrodes from Kerr electro-optic measurements. The method accurately reconstructed the electric field from numerically generated data. However in the presence of experimental noise the performance was less satisfactory. The measurements were especially noisy and unstable near the needle tip which is also the interesting region since most charge injection initiates here. We develop a new algorithm for Kerr electro-optic reconstruction of space charge in axisymmetric point/plane electrode geometries. The algorithm is built on the finite element method (FEM) for Poisson’s equation and will be called finite element based Kerr electro-optic reconstruction (FEBKER) hereafter. FEBKER calculates the space charge density directly to avoid the numerical problems associated with taking the divergence of the electric field, uses single parameter light intensity measurements to enable transient analysis, which otherwise is difficult since multiple parameter intensity measurements are slow due to the rotation of polarizers, and is capable of reconstruction even when the number and/or position of measurements are limited by the electrodes and/or the experimental setup. The performance of the algorithm is tested on synthetic Kerr electro-optic data obtained for an axisymmetric point/plane electrode geometry in transformer oil with specified space charge density distributions. The impact of experimental error is analyzed by incorporating random error to the synthetic data. Regularization techniques that decrease the impact of experimental error are applied. In principle FEBKER is applicable to arbitrary three-dimensional geometries as well.

1 INTRODUCTION

In HV environments electric field phenomena is governed by Poisson’s equation

\[ \nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon} \]  

(1)

where \( \phi \) is the electric potential, \( \rho \) is the space charge density, \( \varepsilon \) the dielectric permittivity, and \( \vec{r} \) is the position vector. If the space charge distribution is known then (1) can be solved by various numerical methods. The solution can then be used to find the electric field

\[ \vec{E}(\vec{r}) = -\nabla \phi(\vec{r}) \]  

(2)

which is an important quantity in power apparatus design since breakdown of the dielectric materials occurs when the electric field magnitude exceeds certain values.

Unfortunately the physical laws that govern space charge injection and transport are not fully known. Thus \( \rho(\vec{r}) \) in (1) cannot be modeled adequately and designs often take space charge to be zero. Such designs cannot predict where the electric field magnitude may exceed safe maxima in the presence of space charge. Indeed some failures in power apparatus are attributed to unexpected accumulation of space and surface charge which result in spark discharges. Thus it is of great interest to understand or at least empirically model space charge injection and transport in insulating dielectrics. This challenge requires accurate measurement of space charge and electric field distributions in experiments. This paper is a part of our continuing effort to measure electric field distributions in transparent liquid and (isotropic) solid dielectrics using the Kerr electro-optic effect [1-10].

The Kerr electro-optic measurement technique utilizes the applied electric field-induced birefringence (Kerr effect) and is applicable to transparent dielectrics such as transformer oil and nitrobenzene. The method uses polarized light as a probe and thus, unlike the traditional metal probe based methods, is not invasive. Earlier Kerr electro-optic measurements were based on optical fringe patterns and limited to highly birefringent liquid dielectrics such as highly purified water, water/ethylene glycol mixtures [2, 4, 5], and nitrobenzene [1, 11, 12]. With the advent of the ac modulation method [13-15], Kerr electro-optic measurements are also used for weakly birefringent dielectrics, most notably transformer oil [15-19].
Either using fringe patterns or the ac modulation method, Kerr electro-optic measurements are limited to one or two-dimensional geometries such as parallel plane electrodes or two concentric or parallel cylindrical electrodes where the electric field magnitude and direction have been constant along the light path. However, to study charge injection and breakdown phenomena, very high electric fields are necessary (\( \approx 10^7 \text{ V/m} \)) with long electrode lengths; for these geometries large electric field magnitudes can be obtained only with very high voltages. Furthermore, in these geometries the breakdown and charge injection processes occur randomly in space, often due to small unavoidable imperfections on otherwise smooth electrodes. The randomness of this surface makes it difficult to localize the charge injection and breakdown and the problem is complicated because the electric field direction also changes along the light path. To create large electric fields for charge injection at a known location and at reasonable voltages, a point electrode often is used in high voltage (typically >100 kV). Furthermore, in these geometries the breakdown and charge injection processes occur randomly in space, often due to small unavoidable imperfections on otherwise smooth electrodes. The randomness of this surface makes it difficult to localize the charge injection and breakdown and the problem is complicated because the electric field direction also changes along the light path. To create large electric fields for charge injection at a known location and at reasonable voltages, a point electrode often is used in high voltage (typically >100 kV).

In our recent research, we have concentrated on this goal. In particular, we applied the characteristic parameter theory of photoelasticity [20] to Kerr electro-optic measurements to reconstruct the axisymmetric electric field distribution of point/plane electrodes in weakly birefringent transformer oil [8] and in strongly birefringent propylene carbonate [9] stressed by relatively low voltage. In both papers we used the ac modulation method and the onion peeling algorithm proposed by Aben [21]. In this algorithm axisymmetric geometries are discretized using planes perpendicular to the axis of symmetry, and the planes are further discretized using annular rings and the electric field distribution is reconstructed layer by layer from outside to inside, using data available from Kerr electro-optic measurements. More recently we also theoretically explored the possibility of applying the onion peeling method to highly birefringent materials, in particular to nitrobenzene, when stressed by point/plane electrodes [22]. There are also attempts for adaptation of algebraic reconstruction techniques (ART) of scalar tomography to Kerr electro-optic measurements [23-27]. Both the onion peeling method and ART are in their development stage and although reported reconstructions of electric field distributions are encouraging, more research is needed before they can be used to quantify space charge distributions.

In this paper, as an alternative to the onion peeling method, we develop a new algorithm that reconstructs space charge from Kerr electro-optic measurements in weakly birefringent media such as transformer oil. The algorithm is built on the finite element method (FEM) for Poisson's equation and will be called finite element based Kerr electro-optic reconstruction (FEBKER). FEBKER incorporates the continuity and irrationality of the electric field into the reconstruction process, calculates the space charge density directly to avoid the numerical problems associated with taking the divergence of the electric field using Gauss's law, and is capable of using traditional regularization techniques. All these properties are expected to reduce the impact of noise on the quality of the reconstructed solutions. In addition FEBKER can be based on single parameter intensity measurements to enable transient analysis which otherwise is difficult since multiple parameter intensity measurements used by the onion peeling method are slow due to the rotation of polarizers. Finally the new algorithm is capable of reconstruction even when the number of measurements are limited by the window size of the experimental setup, certain measurement angles are blocked by the electrodes, or measurement in certain positions are very unreliable to be used, such as the needle tip in point/plane electrodes.

FEBKER is applicable to arbitrary three-dimensional geometries. However, for this paper we only implement the algorithm for axisymmetric geometries and test it on a point/plane electrode geometry. The main complexity in implementation of three-dimensional geometries lies in the mesh generation (see Section 4.3). Since in the near future FEBKER is expected to be used mainly for the point/plane electrode geometries and most practical point/plane geometries are axisymmetric, three-dimensional implementation is left to future work. As we progress through the paper we do note how axisymmetric implementation generalizes to the three-dimensional case.

## 2 KERR ELECTRO-OPTIC MEASUREMENTS

Kerr electro-optic measurements utilize applied electric field induced birefringence. A typical experimental setup is illustrated in Figure 1. The polarizer and the quarter-wave plate are used to obtain...
When circularly polarized light propagates through the HV stressed dielectric, it becomes elliptically polarized due to birefringence. Light intensity measurements at the output provide information on the electric field.

![Figure 1](image1)

**Figure 1.** A basic Kerr electro-optic measurement setup. RSM denotes rotating stepper motor. The main chamber is filled with a liquid dielectric and houses an electrode system (here point/plane electrodes) whose electric field and space charge distribution are under investigation. An advanced setup also includes filtration, vacuum and temperature control systems to control and monitor the dielectric state. Two-dimensional data is obtained either by mechanically moving the laser or expanding the light beam and using a camera or CCD detector at the output.

![Figure 2](image2)

**Figure 2.** The coordinate system used to describe the Kerr effect. Unit vectors \( \hat{m} \) and \( \hat{n} \) are perpendicular to the light propagation direction \( \hat{s} \) with which they form a Cartesian reference frame. Kerr electro-optic birefringence depends only on \( \vec{E}_T \), the transverse component of the applied electric field \( \vec{E} \). Unit vectors \( \hat{e}_1 \) and \( \hat{e}_2 \) are respectively in the direction of \( \vec{E}_T \) and perpendicular to \( \vec{E}_T \) on the mp plane. The counterclockwise angle from \( \hat{m} \) to \( \vec{E}_T \) is denoted by \( \varphi \).

Kerr electro-optic birefringence depends quadratically on the applied electric field magnitude and often is expressed in terms of the refractive index tensor components

\[
\Delta n = n_\parallel - n_\perp = \lambda B E_T^2.
\]

Here \( E_T \) is the magnitude of \( \vec{E}_T \), the component of the applied electric field transverse to the light propagation direction as shown in Figure 2; \( n_\parallel \) and \( n_\perp \) are respectively the refractive indices for light polarized in the direction of \( \vec{E}_T \) and in the direction perpendicular to \( \vec{E}_T \) and the light propagation direction; \( \lambda \) is the free space wavelength of the light; and \( B \) is the material-specific Kerr constant.

For one or two-dimensional geometries such that the applied electric field direction is constant along the light propagation direction, (3) can be integrated directly to find the birefringence induced phase shift \( \Phi \) between the light electric field components parallel and perpendicular to \( \vec{E}_T \)

\[
\Phi = \int_0^l \frac{2\pi}{\lambda} \Delta n \, ds = 2\pi B E_T^2 l
\]

Here \( s \) is the position coordinate along the light path as shown in Figure 2 and \( l \) is the optical path length inside the electrode geometry for which the electric field is assumed constant along \( s \). Thus if \( \Phi \) is measured, the transverse electric field magnitude follows as

\[
E_T = \sqrt{\frac{\Phi}{2\pi B l}}
\]

Typically the light propagation direction is chosen to be along the infinite axis of the one or two-dimensional geometry; \( \vec{E}_T = \vec{E} \). Most past Kerr electro-optic measurements are based on (5) where the optical phase shift \( \Phi \) is measured for one or two-dimensional geometries to yield the electric field magnitude.

When the electric field direction and magnitude vary along the light path, (3) indicates an anisotropic media. Analysis of light propagation in inhomogeneous anisotropic media is in general difficult. However for Kerr electro-optic measurements the birefringence is small

\[
\Delta n \ll 1
\]

and it is possible to use the so called slowly varying amplitude approximation to relate the applied electric field to the light polarization. A detailed forward theory of Kerr electro-optic measurements has been presented in our earlier work [8] which showed that under the condition

\[
\pi B \int E_T^2(s) \, ds = \pi B \int \left[ E_m^2(s) + E_p^2(s) \right] \, ds \ll 1
\]

each Kerr electro-optic measurement can at most yield two characteristic parameters \( \alpha \) and \( \gamma \) which are related to the applied electric field by

\[
\gamma \cos 2\alpha = \pi B \int E_T^2(s) \cos 2\varphi(s) \, ds
\]

\[
\gamma \sin 2\alpha = \pi B \int E_T^2(s) \sin 2\varphi(s) \, ds
\]

\[
\gamma \cos 2\alpha = \pi B \int \left[ E_m^2(s) - E_p^2(s) \right] \, ds
\]

\[
\gamma \sin 2\alpha = \pi B \int 2E_m(s)E_p(s) \, ds
\]

Here \( s_{in} \) and \( s_{out} \) are respectively the entrance and emergence points of the light beam into and out of the HV stressed dielectric, \( \hat{s} \) is the light propagation direction.
propagation direction which with the mutually perpendicular directions \( \hat{m} \) and \( \hat{p} \) form a right-handed (mps) coordinate frame as shown in Figure 2.

Note that (3) and (6) do not in general imply (7). In particular for highly birefringent materials like nitrobenzene (\( B \approx 3 \times 10^{-12} \text{ m/V}^2 \)) under most experimental conditions, (6) holds while (7) does not. This paper is limited to cases for which both (6) and (7) are valid. For materials with small Kerr constants (\( B < 10^{-13} \text{ m/V}^2 \)), in particular for the most common HV insulating transformer oil (\( B \approx 2 \times 10^{-15} \text{ m/V}^2 \)), (7) is valid under most experimental conditions. For highly birefringent materials the algorithms developed in this paper are not applicable and a separate line of research is necessary [22].

The characteristic parameters \( \alpha \) and \( \gamma \) are particularly useful because they can be measured by the same polariscope systems which are used traditionally for one or two-dimensional geometries; the input/output intensity relations are modified simply by replacing the optical phase shift \( \Phi \) with \( 2\alpha \gamma \) and the (constant) electric field direction \( \phi \) with \( \alpha \). In particular for the polariscope system shown in Figure 1 the input/output relation is

\[
\frac{I}{I_0} = 1 + \sin 2\gamma \sin (2\alpha - 2\theta_a) \approx \frac{1}{2} + \gamma \sin (2\alpha - 2\theta_a) \tag{9}
\]

Here we assume that the transmission axis of the polarizer and the slow axis of the quarter wave plate make an angle of \( \frac{\pi}{4} \) to obtain circularly polarized light before entering the medium, \( \alpha \) in (8) is only a small perturbation to \( \frac{1}{2} \) and diffusively to measure accurately by a lock-in amplifier tuned respectively to frequencies \( \omega \) and \( 2\omega \).

In theory (9) can be used to measure both \( \alpha \) and \( \gamma \) by making measurements at two or more values of \( \theta_a \). However since \( \gamma \) is small, \( \gamma \sin 2(\alpha - \theta_a) \) in (9) is only a small perturbation to \( 1/2 \) and difficult to measure accurately. To increase measurement sensitivity the ac modulation method is used by which an ac voltage with known radian frequency \( \omega \) is superposed onto a dc voltage; \( v(t) = v_{dc} + v_{ac} \cos \omega t \). Then the potential distribution \( \phi \) in the dielectric is given by

\[
\phi(\vec{r}) = \phi_{dc}(\vec{r}) + \phi_{ac}(\vec{r}) \cos \omega t \tag{10}
\]

where \( \phi_{dc} \) and \( \phi_{ac} \) are the dc and ac potentials due to respective applied voltages. If \( \omega \) is high enough, any space charge cannot move significantly over the course of the sinusoidal cycle at angular frequency \( \omega \), and does not affect the ac part of the potential [8]. Then (1) separates into two independent equations

\[
\nabla^2 \phi_{dc}(\vec{r}) = 0 \tag{11}
\]

\[
\nabla^2 \phi_{ac}(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon} \tag{12}
\]

and the presence of the ac potential does not effect the dc field-induced space charge injection and transport phenomena under investigation. The ac and dc electric field distributions are respectively given as

\[
\vec{E}_{ac}(\vec{r}) = -\nabla \phi_{ac}(\vec{r}) \tag{13}
\]

\[
\vec{E}_{dc}(\vec{r}) = -\nabla \phi_{dc}(\vec{r}) \tag{14}
\]

To find the measured intensity in terms of the ac and dc components of the electric field, we first note from Figure 3 that

\[
E_{m}(s) = E_T(s) \cos \phi(s) = E_{T_{dc}}(s) \cos \eta(s) + E_{T_{ac}}(s) \cos \zeta(s) \cos \omega t \tag{15}
\]

\[
E_{p}(s) = E_T(s) \sin \phi(s) = E_{T_{dc}}(s) \sin \eta(s) + E_{T_{ac}}(s) \sin \zeta(s) \cos \omega t
\]

Here \( E_T, E_{T_{dc}}, \) and \( E_{T_{ac}} \) are amplitudes and without loss of generality are taken to be positive. Using (15), \( \gamma \cos 2\alpha \) and \( \sin 2\alpha \) in (8) can be expressed in terms of the ac and dc components of the electric field. Substituting these expressions into (9) results in dc, fundamental frequency, and double frequency harmonic light intensity components

\[
I = I_{dc} + I_{\omega} \cos \omega t + I_{2\omega} \cos 2\omega t \tag{16}
\]

where

\[
I_{dc} \approx \frac{I_0}{2} \tag{17}
\]

\[
\frac{I_{\omega}}{I_{dc}} \approx 4\pi B \int_{s_{cm}}^{s_{st}} E_{T_{ac}}(s) E_{T_{dc}}(s) \sin [\eta(s) + \zeta(s) - 2\theta_a] \, ds \tag{18}
\]

\[
\frac{I_{2\omega}}{I_{dc}} \approx \pi B \int_{s_{cm}}^{s_{st}} E_{T_{ac}}(s) \sin [2\zeta(s) - 2\theta_a] \, ds \tag{19}
\]

Here \( I_{\omega} \) and \( I_{2\omega} \) are respectively the peak amplitudes of the fundamental frequency and double frequency harmonics of the optical signal at the photodetector; Equation (17) is a result of (7) because the electric field dependent dc terms are negligible; in (18) and (19) ratios to \( I_{dc} \) instead of \( I_0 \) are preferred, because \( I_{ac} \) is available at the output where measurements are taken. Both \( I_{dc}/I_{dc} \) and \( I_{2\omega}/I_{dc} \) may be measured accurately by a lock-in amplifier tuned respectively to frequencies \( \omega \) and \( 2\omega \).

To clarify what the measurables are, \( \sin [2\zeta(s) - 2\theta_a] \) in (19) can be expanded using trigonometric identities, \( \cos 2\theta \) and \( \sin 2\theta \) can be pulled out of integrals (\( \theta_a \) is independent of \( s \)) and again using trigonometric identities (19) can be expressed as

\[
\frac{I_{2\omega}}{I_{dc}} = \gamma_{ac} \sin (2\alpha_{ac} - 2\theta_a) \tag{20}
\]

where
Comparison of (21) to (8) shows that \( \gamma_{ac} \) and \( \alpha_{ac} \) are the characteristic parameters that correspond to the space-charge free (Laplacian) electric field distribution \( \mathbf{E}_{ac} \). From (20) light intensity measurements at two or more values of \( \theta_d \) determine \( \gamma_{ac} \) and \( \alpha_{ac} \). Again we stress that if the period \( T = 2\pi/\omega \) is much shorter than the transport time for ions to migrate significant distances over the course of a sinusoidal angle, the ac charge density is essentially zero and \( \gamma_{ac} \) is described by solutions to Laplace’s equation in (11) and (13).

Similar to (19) Equation (18) can also be manipulated to obtain

\[
\frac{I_d}{I_{dc}} = \gamma_{hy} \sin (2\alpha_{hy} - 2\theta_d) \tag{22}
\]

where

\[
\gamma_{hy} \cos 2\alpha_{hy} = 4\pi B \int_{\sigma_{hy}}^{a_{hy}} E_{Ty}(s) E_{Tk}(s) \cos \gamma(s) + \xi(s) ds
\]

\[
= 4\pi B \int_{\sigma_{hy}}^{a_{hy}} E_{ma}(s) E_{pa}(s) ds
\]

\[
- 4\pi B \int_{\sigma_{hy}}^{a_{hy}} E_{pa}(s) E_{Pu}(s) ds
\]

\[
\gamma_{hy} \sin 2\alpha_{hy} = 4\pi B \int_{\sigma_{hy}}^{a_{hy}} E_{Ty}(s) E_{Tk}(s) \sin \gamma(s) + \xi(s) ds
\]

\[
= 4\pi B \int_{\sigma_{hy}}^{a_{hy}} E_{ma}(s) E_{pa}(s) ds
\]

\[
+ 4\pi B \int_{\sigma_{hy}}^{a_{hy}} E_{ma}(s) E_{pa}(s) ds
\]

Using (22), light intensity measurements at two or more values of \( \theta_d \) determine \( \gamma_{hy} \) and \( \alpha_{hy} \). Here ‘hy’ refers to hybrid, and indicates that the characteristic parameters \( \gamma_{hy} \) and \( \alpha_{hy} \) depend both on the ac and the dc fields.

Equations (20) to (23) are the Kerr electro-optic measurement expressions that relate measurable \( \alpha_{ac} \), \( \alpha_{hy} \), \( \gamma_{ac} \) and \( \gamma_{hy} \) to applied electric field components when the ac modulation method is used. As a solution to Laplace’s equations, the ac potential \( \phi_{ac} \) can be found by traditional numerical methods. Thus the measurement of \( \alpha_{ac} \) and \( \gamma_{ac} \) does not really provide any new information on the ac electric field. Such measurements however are useful for experimental setup calibration and can be used potentially in statistical analysis of the measurement sensitivity and error. On the other hand, the dc potential \( \phi_{dc} \) and electric field \( \mathbf{E}_{dc} \) cannot be found by traditional numerical methods since the space charge distribution \( \rho \) is not known. The reconstruction algorithm FEBKER developed in this paper reconstructs \( \rho \) in (12) from a set of measurements of \( \gamma_{hy} \) and \( \alpha_{hy} \) which are related to the applied electric field \( \mathbf{E}_{dc} \) and numerically available \( \mathbf{E}_{ac} \) in (23). The dc electric field and the potential of course are related in (14).

One important property of (23) that makes the development in this paper possible is its linearity in \( \phi_{dc} \). Equation (23) depends on the dc electric field components linearly and ac electric field components enter the integral relations in (23) only as known (numerically available) weight functions. Thus the ac modulation method not only increases the sensitivity of Kerr electro-optic measurements but also linearizes the intensity-electric field relations to enable FEBKER.

Also note that in (23) the magnitude of \( \gamma_{hy} \) is directly proportional to the applied ac voltage \( V_{ac} \). Thus it is advantageous to use a large ac voltage to increase the magnitude of \( \gamma_{hy} \) and decrease the impact of white noise on measurements. On the other hand, applying a very large ac voltage can influence the dc phenomena under investigation by causing local breakdowns as combined dc and ac potentials may exceed the breakdown potential. In our recent work \[8,9\] we typically used a dc to ac applied voltage ratio of two to one which was dictated by white noise levels and limitations of our hv amplifier. In our future experimental work we expect to repeat experiments with varying dc to ac applied voltage ratios; ratios as small as ten to one and as large as 1 to 1 are planned.

### 3 Electrode Geometry

FEBKER is applicable to arbitrary 3-dimensional geometries. However at this time it is implemented only for axisymmetric geometries and in this paper it is tested on an axisymmetric point/plane electrode geometry. Here we introduce this test geometry which is also used for descriptive purposes while we develop FEBKER.

![Figure 4. The axisymmetric point/plane electrode geometry used as a case study in this paper.](image-url)
distance of 2.5 mm and tip radius curvature of $R_c = 0.5$ mm. The tip is a hyperboloid of revolution whose equation is $H(r, z) = 1$, where

$$H(r, z) = \frac{(z + 2.5)^2}{d_n^2} - \frac{r^2}{d_n R_c} = 1$$  \tag{24}$$

with $d_n = 5$ mm and $R_c = 0.5$ mm. In (24) we assume that $z$ and $r$ are specified in mm. A cylindrical ground electrode of 10 mm radius bounds the geometry on the sides and the bottom. The dotted line on the top represents an artificial Neumann boundary consistent with the geometry as the upper part of the needle and the surrounding ground form a concentric cylindrical geometry whose electric field distribution is essentially radial.

The radius of the surrounding ground is chosen so that the left and right boundaries are sufficiently larger than the tip/plane distance and thus do not appreciably affect the electric field between the tip and the plane, but small enough so that an excessive numerical load is avoided in the development of FEBKER. In actual experiments the radius of the surrounding ground (often the chamber itself, see Figure 1) is much larger. However, the electric field distributions of such cases do not show any notable characteristic differences from the electric field distribution of the geometry of Figure 4 except a slower rate of spatial decay for electric field components. The 0.5 mm radius of curvature is indicative of those used at MIT for Kerr electro-optic measurements [28]. It is relatively larger than typical values used in dielectric research, and chosen to avoid experimental difficulties related to the very divergent electric field distributions of smaller radius of curvature needles in this development stage of Kerr electro-optic measurement for arbitrary geometries.

The two forms of space charge densities considered for theoretical case studies are illustrated in Figure 5. For constant $r$ the space charge densities linearly increase from their values at ground to a maximum at the needle tip height at $z = 2.5$ mm, and then decrease linearly at a slower rate until reaching the needle electrode. For constant $z$, in the first form the space charge density linearly decreases from its maximum value at $r = 0$ mm to 0 at $r = 1$ mm. The equation of this space charge distribution is

$$\rho(r, z) = \begin{cases} 
\varepsilon_r \rho_0 [1 - \frac{z^2}{d_n^2}] \frac{1 + \frac{z}{d_n}}{d_n} & r < \Delta, z < d \\
\varepsilon_r \rho_0 [1 - \frac{z^2}{d_n^2}] \frac{5 - \frac{z}{d_n}}{\Delta} & r < \Delta, z > d \\
0 & r > \Delta \\
0 & H(r, z) > 1 \quad |r| < r_n
\end{cases}$$  \tag{25}$$

where $d = 2.5$ mm is the tip/plane distance, $\Delta = 1$ mm is the radial extent of the space charge distribution, $\rho_0 = \rho(0, 0)/\varepsilon_r = 0.12$ C/m$^3$, $r_n = 2.7$ mm is the radius of the needle, and $H(r, z)$ is defined in (24). The fourth line in (25) corresponds to the points inside the needle where the charge density is taken to be zero. In the second form the space charge density quadratically decreases from its maximum value at $r = 0$ mm to 0 at $r = 1$ mm so that the space charge distribution is

$$\rho(r, z) = \begin{cases} 
\varepsilon_r \rho_0 [1 - \frac{z^2}{d_n^2}] \frac{1 + \frac{z}{d_n}}{d_n} & r < \Delta, z < d \\
\varepsilon_r \rho_0 [1 - \frac{z^2}{d_n^2}] \frac{5 - \frac{z}{d_n}}{\Delta} & r < \Delta, z > d \\
0 & r > \Delta \\
0 & H(r, z) > 1 \quad |r| < r_n
\end{cases}$$  \tag{26}$$

Both space charge distributions are chosen for their simplicity and reasonable for phenomena where charge is injected from the needle. One important reason for the extensive use of a point/plane electrode geometry in HV research is to localize the charge injection physics near the tip and the space charge densities in (25) and (26), which are non-zero only within a 1 mm radius around the needle tip axis, simulate this localization. Note that the actual value of the relative permittivity $\varepsilon_r$ is never needed since it cancels out in the Poisson equation (1).

The geometry in Figure 4 has smaller radial extent than most point plane electrode geometries. However for the space charge distributions considered in (25) and (26), the electric field components for $r > 5$ mm essentially are equal to the Laplacian (space charge free) electric field distribution; the radial extent is large enough so that for large $r$ the

Figure 5. Case study space charge density distribution for the geometry shown in Figure 4. The two-dimensional charge densities $\rho(r, z)$ can be found by multiplying the $z$ dependence on the left and the $r$ dependence shown on the right. Here $\varepsilon_r$ is the relative permittivity of the medium.

Figure 6. Electric field component plots above the ground plane for $z = 2.25$ mm with $\rho_0$ values of 0, 0.04, 0.08, 0.12, 0.16 and 0.20 C/m$^3$ as the parameter. The arrows indicate the direction of increasing $E_v$. Notice that all plots are essentially identical for $r > 5$ mm.
electric field distribution reduces to the Laplacian one. This is illustrated in Figure 6 where we plot the electric field components when the space charge distribution is given by (25). Here we also vary \( \rho_0 \) to see the effects of larger space charge density magnitudes. Note that in these plots \( \rho_0 = 0.2 \text{ C/m}^3 \) is chosen as the largest magnitude as for \( \rho_0 = 0.24 \text{ C/m}^3 \) inversion occurs; the electric field changes direction on the needle tip. Figure 6 illustrates that the geometry in Figure 4 is adequate to understand the general characteristics of the electric field distributions of similar geometries with larger plane electrodes and/or large radial extent. The geometry is certainly practical and can be used for optical measurements using conductive coated glass as the surrounding ground electrode.

![Figure 7. A Kerr electro-optic measurement path (left) is projected onto the rz plane (right). In this paper all measurements are taken from optical rays that are perpendicular to the axis of symmetry z. Assuming the right figure also corresponds to the \( s=0 \) plane in the \( msp \) coordinate frame, the first point of the projected path completely describes the measurement in the \( msp \) plane. For each point along the light path on the left the applied electric field component values \( (E_m, E_p, \text{and } E_r) \) can be found from the applied electric field component values \( (E_x \text{ and } E_z) \) at the corresponding point on the projection (right).](image)

In the rest of this paper we develop an algorithm that reconstructs the space charge density distributions from Kerr electro-optic measurements and test it on the point/plane geometry introduced in this Section. For the tests we limit the measurements to be perpendicular to the axis of symmetry \( z \) as illustrated in Figure 7 (\( s = 0 \)). We choose the \( msp \) coordinate system such that \( \vec{m} = \vec{z} \) and its origin coincides with the origin of the \( rz \) plane. Then each measurement can be identified by its location on the \( msp \) plane as illustrated by a cross in Figure 7. Notice that a distinction between \( p \) and \( r \) is necessary; along the light path shown in Figure 7 (left) \( p \) is constant but \( r \) varies and \( \vec{r} \) coincides with \( \vec{p} \) only at the \( s = 0 \) plane. No such distinction is necessary between \( m \) and \( z \). For these choices (21) and (23) can be expressed more conveniently as radial integrals

\[
\gamma_{nc} \cos 2\alpha_{nc} = 2\pi B \int_{p}^{r_{nz}} \frac{E^2_{zn}(r) - \frac{p^2}{r^2} E^2_{zn}(r)}{\sqrt{r^2 - p^2}} \frac{r \, dr}{\sqrt{r^2 - p^2}} \tag{27}
\]

\[
\gamma_{nc} \sin 2\alpha_{nc} = 2\pi B \int_{p}^{r_{nz}} \frac{2p}{r} E_{zn}(r) E_{zn}(r) \frac{r \, dr}{\sqrt{r^2 - p^2}} \tag{28}
\]

and

Here the integration is along the projection of the light path on the \( rz \) plane and \( p \) refers to the coordinate of the location of the measurement. The projection is illustrated in Figure 7 (right). Since the electric field distribution is axisymmetric, the applied electric component values at each point along the light path in Figure 7 (left) can be found from the (axisymmetric) electric field components at the corresponding point on the projection in Figure 7 (right) enabling transformation of (21) and (23) into (27 and (28). For axisymmetric implementations in this paper (28) is used instead of the general form (23).

4 DISCRETIZATION AND FEM

4.1 TRIANGULAR DISCRETIZATION

Like the finite element method (FEM), FEBKER requires discretization of the electrode geometry to approximate the local continuous field quantities electric potential, electric field, and space charge density, by a finite number of parameters. In particular for axisymmetric geometries, to which the implementation in this paper is limited, most FEM implementations use triangle and/or quadrilateral discretizing elements of the \( rz \) plane. In this work we only use triangles.

![Figure 8. Two views of the mesh used for finite element solutions for the geometry shown in Figure 4.](image)

The finite element mesh used for the test geometry is shown in Figure 8. Since the electrode geometry is axisymmetric only one half needs to be discretized. This is a Delaunay graded mesh, generated by our previously developed mesh generator [29], that uses the algorithm in [30]. We refine the mesh around the needle extensively to minimize the numerical errors from the FEM solutions especially for the specified space charge distributions of Figure 5. There are 5377 triangles in the discretization.
For arbitrary three-dimensional geometries the triangles often are replaced by tetrahedrals. The implementation of three-dimensional mesh generators are complex and commercially/academically available replaced by tetrahedrals. The implementation of three-dimensional mesh generators usually are not flexible enough to be used in this kind of research where extensive numerical experiments are necessary. In fact this is the main reason that in this paper we limit the implementation of FEBKER to axisymmetric geometries. Beside the mesh generation, implementation of FEBKER for three-dimensional geometries presents no difficulty.

4.2 DISCRETIZATION OF FIELD QUANTITIES

Once the electrode geometry is discretized, the simplest approach to approximate any particular continuous quantity is to assume it is stepwise constant within each triangle. However as discussed in Section 4.3, to find numerical solutions to Poisson's equation, the FEM postulates and minimizes an integrated error criterion which depends on the gradient of the potential. This excludes using step-wise constant potentials within triangles. Instead, first and second order polynomials that interpolate nodal values of potential within triangles are commonly used; second order polynomials are often preferred to make the electric field continuous from triangle to triangle. In this work we employ second order polynomials to discretize the electric potential distribution. The electric field components are then found by taking the gradient of the potential resulting in first order polynomials.

The interpolating polynomials within a triangle are best expressed with the so called triangular coordinates which are first order polynomials that are associated with the vertices of the triangle. There are three triangular coordinates \( \zeta_i \); one for each vertex \( i = 1, 2, 3 \) \[ \zeta_i(r, z) = A_i r + B_i z + C_i \] (29) where \( A_i, B_i, C_i \) are the solutions of the matrix equations

\[
\begin{bmatrix}
\tilde{r}_i & \tilde{z}_i & 1 \\
\tilde{r}_j & \tilde{z}_j & 1 \\
\tilde{r}_k & \tilde{z}_k & 1
\end{bmatrix}
\begin{bmatrix}
\hat{A}_i \\
\hat{B}_i \\
\hat{C}_i
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (30)

Here \( (i, j, k) \) are the even permutations of \( (1, 2, 3) \); the coordinates \( \tilde{r}_i \) and \( \tilde{z}_i \) refer to the vertex \( i \) of the triangle as shown in Figure 9; and we use tilde to denote local quantities defined for each triangle. The first row in (30) states that \( \zeta_i \) takes the value 1 on the vertex \( i \) and the second and third rows state that \( \zeta_i \) takes the value 0 on vertices \( j \) and \( k \). In fact, using these properties the polynomial that interpolates the nodal values of potential on the vertices \( \phi_1, \phi_2, \phi_3 \) can be written in terms of triangular coordinates by inspection

\[
\phi(r, z) = \tilde{\phi}_1 \zeta_1(r, z) + \tilde{\phi}_2 \zeta_2(r, z) + \tilde{\phi}_3 \zeta_3(r, z) \] (31)

A second order polynomial has 6 coefficients for \( r, r^2, z, z^2, rz \) and 1, thus in addition to the three vertices three more nodal values are needed for interpolation; the midpoints of each edge shown in Figure 9 are most frequently used. By manipulating (29) and (30) it can be shown that \( \zeta_1 \) is identically 0 on edge 1, \( \zeta_2 \) varies from 1 to 0 and \( \zeta_3 \) varies from 0 to 1 as one moves from node 2 to node 3, and on node 4 both \( \zeta_2 \) and \( \zeta_3 \) take the value 0.5. Similar properties hold for the other edges and using these properties the polynomial that interpolates the nodal values of potential \( \phi_i \) \( (1 \leq i \leq 6) \) again can be written in terms of triangular coordinates by inspection

\[
\phi(r, z) = \tilde{\phi}_1 \zeta_1(r, z) [2 \zeta_1(r, z) - 1] + 4 \tilde{\phi}_2 \zeta_2(r, z) [2 \zeta_2(r, z) - 1] + 4 \tilde{\phi}_3 \zeta_3(r, z) [2 \zeta_3(r, z) - 1] + 4 \tilde{\phi}_4 \zeta_4(r, z) [2 \zeta_4(r, z) - 1] + 4 \tilde{\phi}_5 \zeta_5(r, z) [2 \zeta_5(r, z) - 1] + 4 \tilde{\phi}_6 \zeta_6(r, z) [2 \zeta_6(r, z) - 1] \] (32)

Equation (32) describes the second order interpolating polynomials that are used to approximate the potential distribution \( \phi(r, z) \) within each triangle. Totality of these interpolating polynomials is the overall approximation used to discretize potential distribution \( \phi(r, z) \). We denote this approximation by \( \phi_a(r, z) \) (a for approximate). Since \( \phi_a(r, z) \) is determined by the nodal values of the potential, it can be described by a column vector \( \Phi \) whose elements are the nodal values of the potential on the discretization points; the vertices of the triangles and the midpoints of the edges. To go from \( \Phi \) to \( \phi_a(r, z) \), the triangle in which \( (r, z) \) lies is found and the approximate value of the potential is determined by evaluating (32) at \( (r, z) \) using the six nodal values given as the entries of \( \Phi \). Of course except those corresponding to the nodes on the electrodes with known voltages, the values of the entries of \( \Phi \) are not known which is the reason the potential is discretized in the first place.

Notice that the use of second order polynomials increases the size of \( \Phi \) increasing the size of the FEM problem (the number of unknowns). On the other hand within a triangle a second order polynomial approximates the electric potential better than a first order polynomial and thus solving this increased sized problem also results in increased level of accuracy on the potential solutions; typically the accuracy level achieved by using second order polynomials is comparable to the level of accuracy achieved by first order polynomials and a finer mesh (increased number of unknowns). In this work we prefer second order polynomials to make the electric field continuous so that the divergence can be calculated numerically. Calculation of the divergence of the Laplacian electric field provides a measure for the discretization error introduced by the finite element mesh. This measure is particularly suited for this work where the main quantity of interest is the space charge.

From the discretized potential the electric field within each triangle is found by taking the gradient of (32)
\[ E_r(r, z) = -\frac{\partial \phi(r, z)}{\partial r} \]
\[ E_z(r, z) = -\frac{\partial \phi(r, z)}{\partial z} \]  
resulting in first order polynomials. Notice from (32) and (33) that the electric field is also described in terms of the nodal values of the potential.

For a typical FEM problem, the space charge distribution \( \rho(r, z) \) is known and in fact does not require to be discretized. However to avoid space charge density specific implementations \( \rho(r, z) \) is often discretized in terms of the mesh that is used to discretize the potential. In this work where the charge density is not known, it is not only convenient but also necessary to use a discretization. We discretize the space charge distribution by assuming it is step wise constant within each triangle. Similar to the potential the resulting approximation \( \rho_u(r, z) \) can also be described by a column vector \( \mathbf{p} \) whose elements are the values of the space charge density distribution within each triangle.

### 4.3 THE FINITE ELEMENT METHOD

The FEM is one of the most widely used numerical methods for solution of Poisson’s equation. There are many excellent books on the subject. In particular [31] is a very easy to read book and was frequently consulted for this work. For a more detailed treatment we refer the reader to the book by Hughes [32]. Here we give a brief overview.

The two basic steps of FEM are the postulation of a trial electric potential solution in terms of undetermined parameters and determination of these parameters by some error criterion. The first step is discussed already in Sections 4.1 and 4.2, where the trial solutions are described in terms of interpolating polynomials and the undetermined parameters are the elements of \( \Phi \) except those that correspond to nodes on the electrodes. To this end we partition \( \Phi \) into two parts

\[ \Phi = \begin{bmatrix} \Phi_{u} \\ \Phi_{d} \end{bmatrix} \]  
(34)

Here \( \Phi_u \) and \( \Phi_d \) are themselves column vectors where \( \Phi_d \) (d stands for Dirichlet) constrains the values that correspond to the nodes on the electrodes and \( \Phi_u \) contains the others (u stands for unknown). Boundaries on which the potential is known are called the Dirichlet boundaries and the corresponding boundary condition is called a Dirichlet boundary condition. By requiring the values on the nodes that reside on the electrodes to be that of the applied voltages in (34), Dirichlet boundary conditions are directly imposed on the trial solution \( \Phi \). Note that the artificial Neumann boundary condition shown in Figure 4 and the symmetry condition on the axis of symmetry are natural boundary conditions [31]; their explicit imposition is not required in our formulation.

Once the trial potential solutions are postulated, the unknown parameters \( \Phi_u \) are to be found such that the resulting solution is as close to the actual solution of Poisson’s equation as possible. Various error functionals are in use to measure the closeness of a numerical solution. In this work we use the simple energy functional for a Poissonian system which is given as

\[ \mathcal{L}[\phi, \rho, V] = \iiint_V \left( \frac{1}{2} \varepsilon |\nabla \phi(r)|^2 - \rho(r) \phi(r) \right) dV \]  
(35)

or for axisymmetric geometries using cylindrical coordinates

\[ \mathcal{L}[\phi, \rho, A] = \int_A 2\pi r \left\{ \frac{1}{2} \varepsilon \left( \frac{\partial \phi(r, z)}{\partial r} \right)^2 + \frac{1}{2} \varepsilon \left( \frac{\partial \phi(r, z)}{\partial z} \right)^2 - \rho(r, z) \phi(r, z) \right\} dr dz \]  
(36)

where \( A \) is the area of the two-dimensional projection of the solution region in the \( rz \) plane. The minimum energy principle states that the solution of the Poisson equation minimizes the energy function (35). Thus if \( \Phi_u \) is chosen such that it minimizes \( \mathcal{L}[\phi_a, \rho_a, A] \) then \( \phi_a \) is as close to the actual solution \( \phi \) as possible in the total energy sense. Here \( \rho_a \) denotes the step-wise constant space charge density described in Section 4.2.

To do the minimization with respect \( \Phi_u \), first \( \mathcal{L}[\phi_a, \rho_a, A] \) is expressed as a sum of integrals over the triangles

\[ \mathcal{L}[\phi_a, \rho_a, A] = \sum_e \mathcal{L}[\phi_e, \rho_e, A_e] \]  
(37)

where \( e \) ranges over the triangles and \( A_e, \phi_e, \rho_e \) respectively denote the area, the local interpolating polynomial defined in (32) and the constant space charge density, all referring to triangle \( e \). For each triangle, \( \mathcal{L}[\phi_e, \rho_e, A_e] \) can be evaluated in terms of the nodal values by substituting (32) and carrying out the integration analytically which is possible since for each triangle the spatial dependence is known in terms of the triangular coordinates and \( \rho_e \) is constant. Since \( \mathcal{L}[\phi_e, \rho_e, A_e] \) depends on the square of the gradient of the potential (see (35) or (36)), each \( \mathcal{L}[\phi_e, \rho_e, A_e] \) is a quadratic form in terms of nodal values of the potential in triangle \( e \). The overall sum in (37) is then a quadratic form in terms of \( \Phi \) which can be expressed in matrix form as

\[ \mathcal{L}[\phi_a, \rho_a, A] = \frac{1}{2} \Phi^T P \Phi - \Phi^T T \rho \]  
(38)

Here \( P \) and \( T \) are the matrices that result when the sum in (37) is evaluated analytically in terms of the nodal values of the potential. Each node contributes to the term \( e \) in the sum (37) only if it belongs to triangle \( e \). As a result \( P \) and \( T \) are sparse. It follows from the symmetry of (32) with respect to the nodal values that \( P \) is symmetric. Furthermore since for zero space charge density \( \mathcal{L}[\phi_a, \rho_a, A] \) must always be positive (see (35) or (36)) \( P \) is positive-definite.

To minimize \( \mathcal{L}[\phi_a, \rho_a, A] \) in terms of \( \Phi_u \), the derivative is taken with respect to each entry of \( \Phi_u \) and equated to zero resulting in a square matrix equation of the size of \( \Phi_u \)

\[ \mathbf{P}_{uu} \Phi_u = \mathbf{T}_u \rho - \mathbf{P}_{ud} \Phi_d \]  
(39)

where \( \mathbf{P}_{uu}, \mathbf{P}_{ud}, \) and \( \mathbf{T}_u \) are parts of \( \mathbf{P} \) and \( \mathbf{T} \) when they are partitioned with respect to the unknown and Dirichlet nodal values.

Equation (39) is the final equation of the FEM. In the usual FEM problem \( \rho \), and thus the right hand side, is known and (39) yields \( \Phi_u \) resulting in the approximate solution \( \phi_a (r, z) \). Since \( \mathbf{P}_{uu} \) is a sparse, symmetric and positive-definite matrix, solution of the matrix equation (39) for \( \Phi_u \) poses no difficulty.
5 DISCRETIZED MEASUREMENT EXPRESSIONS

In our work \( \rho \) is not known and thus (39) does not yield \( \Phi_u \) but serves as a discrete Poisson equation that relates the potential and the space charge density. The main idea of this paper is to relate \( \Phi_u \) to a set of Kerr electro-optic measurements and augment (39) with this information to obtain \( \rho \).

When the ac modulation method is used for sensitive Kerr electro-optic measurements, the potential is composed of dc and ac parts as described in Section 2. Then the approximate electric potential distribution takes the form

\[
\Phi = \Phi_{dc} + \Phi_{ac} \cos \omega t \tag{40}
\]

The ac and dc parts respectively satisfy the Laplace and Poisson equations. The unknown entries of the ac potential directly follow from (39) with \( \rho = 0 \)

\[
\Phi_{u,ac} = -\frac{D}{\epsilon} P_{udc} \Phi_{udc} \tag{41}
\]

Equation (39) however cannot be used for the dc potential since \( \rho \) is not known and in fact is the quantity to be determined by Kerr electro-optic measurements. To incorporate Kerr electro-optic data into (39), development of relations between a set of Kerr electro-optic measurements and the approximate ac and dc potential distributions is necessary.

For each measurement position in Figure 10 the corresponding Kerr electro-optic measurement is related to the applied electric field in (28). The light path can be projected onto the finite element mesh as illustrated in Figure 10 (also see Figure 7). Notice that the projected path is partitioned into smaller segments by the triangles. Each integral in (28) can be expressed as a sum of integrals over these elemental segments. These integrals can then be evaluated analytically in terms of nodal values of the potential substituting (32) and (33) in (28). Evaluated integrals are quadratic forms in nodal ac and dc potential values and can be written in matrix form as

\[
k_{v,ac} = \frac{\gamma_{ac} \cos 2\alpha_{ac}}{\pi B} = \Phi_{ac}^T M_{ac}^{(i)} \Phi_{dc}
\]

\[
k_{v,ac} = \frac{\gamma_{ac} \sin 2\alpha_{ac}}{\pi B} = \Phi_{ac}^T M_{ac}^{(i)} \Phi_{ac}
\]

\[
k_{v,by} = \frac{\gamma_{by} \cos 2\alpha_{by}}{4\pi B} = \Phi_{by}^T M_{by}^{(i)} \Phi_{dc}
\]

\[
k_{v,by} = \frac{\gamma_{by} \sin 2\alpha_{by}}{4\pi B} = \Phi_{by}^T M_{by}^{(i)} \Phi_{by}
\]

where we index the measurements by \( i \) and the entries of the matrices \( M_{ac}^{(i)} \) and \( M_{by}^{(i)} \) are the results of analytical evaluations.

Equation (42) is an expression for a single measurement. For a set of measurements like in Figure 10 Equation (42) becomes

\[
k_{v,ac} = \Phi_{ac}^T M_{ac} \Phi_{ac}
\]

\[
k_{v,by} = \Phi_{by}^T M_{by} \Phi_{by}
\]

where the column vectors \( k_{v,ac}, k_{v,by}, k_{c,by}, k_{s,by} \) respectively contain \( k_{v,ac}^{(i)}, k_{v,ac}^{(i)}, k_{c,by}^{(i)}, k_{s,by}^{(i)} \) and the matrices \( M_{ac} \) and \( M_{by} \) contain \( M_{ac}^{(i)} \) and \( M_{by}^{(i)} \). For example, in its explicit form (43) is

\[
k_{v,by} = \left[ \begin{array}{c}
k_{v,by}^{(1)} \\
k_{v,by}^{(2)} \\
\vdots \\
k_{v,by}^{(n)}
\end{array} \right] = \Phi_{ac}^T M_{ac} \Phi_{dc}
\]

\[
k_{v,by} = \Phi_{by}^T M_{by} \Phi_{by}
\]

where \( n \) is the number of measurements.

Since \( \Phi_{ac} \) is available from the FEM in (41), \( k_{v,ac} \) and \( k_{v,by} \) in Equation (43), do not provide any new information. The expressions for \( k_{c,by} \) and \( k_{s,by} \) in (43) can be combined into a single matrix equation

\[
k = \left[ \begin{array}{c}
k_{c,by} \\
k_{s,by}
\end{array} \right] = \Phi_{ac}^T M_{ac} \Phi_{dc}
\]

which relate the measurements \( k \) to the dc potential. Since the potential values on the Dirichlet boundaries are known already, we write (45) in the form

\[
M \Phi_{udc} = k - M_{ac} \Phi_{ac}
\]

Here \( M \) and \( M_{ac} \) are the partitioned parts of the matrix in (45) with respect to the unknown and known (Dirichlet) nodal potential values

\[
\left[ \begin{array}{c}
M \\
M_{ac}
\end{array} \right] = \left[ \begin{array}{c}
\Phi_{ac}^T M_{ac} \\
\Phi_{by}^T M_{by}
\end{array} \right]
\]

Figure 10. The left figure illustrates a typical set of measurement positions for which FEBKER is intended. This is also the set used for testing in this paper. Each point relates to the actual light ray as described in Figure 7. There are 517 measurement positions which are uniform with 0.1 mm interspacing. The circle in the middle is used to group measurements closer to the tip and used for illustration in Section 7.2. The right figure shows the projection of the light path of one of the outer positions on the finite element mesh (also see Figure 7).

In Figure 10 we illustrate a typical set of Kerr electro-optic measurements for which FEBKER is intended and is used for the tests in this paper. The set consists of uniformly spaced measurement positions within the 1.5 mm radius of the axis of symmetry with interspacing of 0.1 mm. It is expected that our future Kerr electro-optic measurements will expand the light beam to cover the measurement area and use a CCD light detector for which 0.1 mm resolution is within today's standards. The tip/plane distance of 2.5 mm and 0.1 mm spacing are also reasonable for small beam diameter \( (w \lesssim 1 \text{ mm}) \) measurements on which our previous experiments are based on [8, 28]. For small beam measurements the implementation in this paper can be slightly modified to take into account the finite beam size using an averaging scheme. This will not be investigated in this paper.
Equation (46) is the final result of this Section. It relates the unknown dc potential values to the Kerr electro-optic measurements and the dc applied voltages. In principle it can be used as a basis for a family of reconstruction algorithms which reconstruct the potential distribution from Kerr electro-optic measurements. The potential distribution then can be used to calculate the electric field and space charge density. However with such an approach, any reconstruction error in the potential is strongly magnified in the electric field and space charge density values because of the differentiation, and Equation (39), which in essence is the discretized form of Poisson's equation that relates the electric potential and space charge density, is not utilized.

For arbitrary three-dimensional geometries most of the discussion of this Section remains valid. One can begin with (23) and arrive at an equation identical in form to (46). The set of measurements however is likely to be more extensive and should probably include multiple views, a set with light paths in the same direction may not be enough.

In this paper we calculate \( \Phi_m \) from (41) using the same finite element mesh in Figure 8 that is used to calculate \( M_c \) and \( M_k \). The numerical error in \( \Phi_m \) is considered to be a part of the discretization error in (46) where we relate measurements to discrete dc potential \( \Phi_{dc} \). In general it is not necessary to use the same mesh to calculate \( \Phi_{dc} \); using a finer mesh one easily can decrease the error in \( \Phi_m \) to negligible levels when compared to the experimental measurement error levels in \( k \).

6 FEBKER

The equation that FEBKER relies on follows from (39) and (46)

\[
\mathbf{M}^{-1}_{P_{uu}} \mathbf{T}_u \rho - k = \left[ \mathbf{M}^{-1}_{P_{uu}} \mathbf{P}_{ud} - \mathbf{M}_{d} \right] \Phi_{dc} = 0 \tag{48}
\]

Equation (48) is fundamental to this paper. It relates the unknown space charge distribution \( \rho \) to the measurements \( k \) and the applied dc voltage \( \Phi_{dc} \).

In principle with enough measurements (48) is an overdetermined or a square matrix equation and can be solved as a least squares problem for which a number of established methods exist [33]. However solving (48) directly for \( \rho \) has two potential problems. First \( \mathbf{M}^{-1}_{P_{uu}} \mathbf{T}_u \) is a full matrix for which solution of the least squares problem is computationally expensive. The size of the least squares problem is determined by the size of \( \rho \) and for a typical finite element mesh with thousands of elements the computational time requirement can become unacceptable. The second problem stems from the fact that the FEM, as formulated in this work, minimizes an integrated error criterion. For individual triangular elements there may be large discrepancies between the specified charge density and the charge density found by taking the divergence of the numerical electric field solution. For (48) this translates into a considerable number of erroneous elements in \( \rho \). Of course one can always refine the mesh to improve \( \rho \), however this increases both the size of the problem and the number of measurements to make (48) an overdetermined or a square matrix equation.

Notice that the size of the least square problem in (48) is determined by the number of parameters, step-wise constant values within triangles, that are used to describe the space charge density. In this work we address the problems associated with solving (48) directly by specifying the space charge density distribution with fewer parameters. Reducing the number of parameters is achieved by physical considerations and by specifying the space charge density using a mesh that is less detailed than the one used for FEM formulation.

6.1 PHYSICAL CONSIDERATIONS

The physical expectations that the electrode geometry imposes can be used to reduce the number of parameters that describe the space charge distribution. For the point/plane electrode geometry, which constitutes the case study in this paper, space charge injection is expected to be localized to the needle tip. In fact the wide use of point/plane electrode geometry in HV research is due to the localization of space charge injection. Far from the needle axis the space charge density is expected to vanish. Thus one can assume from the outset that the elements of \( \rho \) that corresponds to outer triangles are zero.

In Figure 11 we illustrate this approach on the test geometry. Here we assume that the space charge density is zero outside a 1.25 mm radius of the axis of symmetry. This region is meshed by uniform squares which are used in Section 6.2 to further decrease the number of parameters that describe the space charge density. The value 1.25 mm is chosen as opposed to the actual value of 1 mm to simulate the fact that in an actual experiment the exact radius is not known but can be estimated using the convergence of fundamental and double harmonic light intensity measurements with the ac modulation method.

For arbitrary geometries, if the charge injection is localized, the method described directly carries over. Otherwise the method is not applicable. Note however that for geometries such that charge injection is not localized, existence of regions such as the tip of the needle where the mesh has to be refined extensively is unlikely. For such geometries the size of the problem in (48) is often smaller in the first place.

In this paper, as a physical consideration, we only use the locality of charge injection to decrease the size of (48). This however is not the only possibility. One can, for example, attempt to postulate a physical model with unknown parameters, possibly in the form of an analytical function, and describe \( \rho \) in terms of these fewer unknown parameters possibly by linearizing the postulated function in small intervals. Still another possibility is to impose positive or negative polarity conditions on the space charge density by physical arguments based on the type of the electrode material. These possibilities are not further pursued in this paper.
6.2 \( \rho \) MESH METHOD

Beside the local charge injection assumption of Section 6.1 we reduce the size of (48) by specifying the space charge density \( \rho \) in terms of a less detailed mesh with fewer number of unknowns. In the following, such a mesh is referred to as \( \rho \)-mesh and the resulting methodology is referred to as the \( \rho \)-mesh method. Figure 11 illustrates the \( \rho \)-mesh used for the test geometry consisting of uniform squares imposed on top of the triangular FEM mesh in the region where the space charge density is assumed to be nonzero.

Recall that each unknown entry in \( \rho \) corresponds to a triangle in the FEM mesh. If we group triangles in \( \rho \)-mesh squares and assume the space charge density is step-wise constant over the \( \rho \)-mesh (as opposed to over the individual triangles) then since each square contains many triangles, the number of unknowns reduces dramatically. Gathering a group of triangles in a \( \rho \)-mesh square on which the space charge density value is assumed to be constant is, in essence, averaging unknown space charge density values of individual triangles into a single space charge density value. Thus, the \( \rho \)-mesh method also averages out the error in the space charge values.

It is important to note that specifying a \( \rho \)-mesh to reduce the number of unknowns in \( \rho \) is not equivalent to formulating the FEM Equation (39) in terms of the \( \rho \)-mesh. One extreme situation which clarifies the distinction is the case when the space charge density is constant throughout the geometry. Then the choice of \( \rho \)-mesh does not affect the quality of reconstructed elements of \( \rho \) from (48), since for a perfect solution the values of the elements should be identical. However if \( \rho \)-mesh also is used to formulate (39) then the accuracy of \( \rho \) will decrease degrading the quality of reconstructed \( \rho \) from (48).

Clearly the spatial resolution of the reconstructed space charge density does depend on the details of the \( \rho \)-mesh. Obviously a reconstruction based on a \( \rho \)-mesh with larger (smaller) squares will have less (more) spatial resolution. The computational load to find the solution with less (more) spatial resolution is also less (more). Thus the detail of a \( \rho \)-mesh is a trade-off between the spatial resolution and the computational load. The number of measurements also figure in the decision for the choice of \( \rho \)-mesh because if the \( \rho \)-mesh is too detailed (48) becomes an underdetermined matrix equation.

We use the uniform square mesh for its simplicity. This does introduce some difficulty in identifying to which square a triangle belongs since the elements of the FEM mesh do not necessarily lie inside a single element in \( \rho \)-mesh. We decide the \( \rho \)-mesh element that an individual triangle belongs to by the position of the triangle centroid; each triangle is assumed to belong to the square that contains its centroid. Although in this paper we only use uniform square \( \rho \) meshes, one can use other kinds of mesh, for example a less detailed triangular mesh.

In a \( \rho \)-mesh it is not necessary to assume that the space charge density is step-wise constant; one can use other assumed solution types. One immediate improvement over the step-wise constant approach is to use local solutions that are linear in \( r \) and \( z \) and interpolate values on the vertices of squares. For a square with edge length \( l \) and lower-left vertex coordinate \((\bar{r}_1, \bar{z}_1)\) such a solution can be expressed by inspection

\[
\hat{\rho}(r, z) = \hat{\rho}_1 \left( 1 - \frac{r - \bar{r}_1}{l} \right) \left( 1 - \frac{z - \bar{z}_1}{l} \right) + \hat{\rho}_2 \left( \frac{r - \bar{r}_1}{l} \right) \left( 1 - \frac{z - \bar{z}_1}{l} \right) + \hat{\rho}_3 \left( \frac{r - \bar{r}_1}{l} \right) \left( \frac{z - \bar{z}_1}{l} \right) + \hat{\rho}_4 \left( 1 - \frac{r - \bar{r}_1}{l} \right) \left( \frac{z - \bar{z}_1}{l} \right)
\]

where \( \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3, \) and \( \hat{\rho}_4 \) are respectively the space charge density values on the lower-left, lower-right, upper-right and upper-left vertices. Each entry of \( \rho \) can then be found in terms of \( \hat{\rho}_i \) by evaluating (49) on the centroid of the corresponding triangle for the square that the triangle belongs to. Notice that although for each square there are now 4 unknowns, most of these unknowns are shared with the neighbor squares. For a typical mesh the number of unknowns defined by the step-wise constant approach is not expected to be dramatically larger than the one defined by a locally linear approach. Any reasonable increase in the unknowns is justified by the superiority that the local linear functions can represent arbitrary space charge distributions. In this paper we use both the step-wise approach and the locally linear approach.

Both the local charge injection approach described in Section 6.1 and the \( \rho \)-mesh method of this section can be mathematically described by a matrix equation that relates \( \rho \) to a new unknown vector which we denote by \( \varrho \)

\[
\rho = U \varrho
\]

Here \( U \) is a rectangular matrix. If the \( U \) row number is larger than its column number then the size of the least squares problem reduces. For the local charge injection approach and step-wise constant \( \rho \) method each row of \( U \) is either all zeros (for triangles outside the \( \rho \)-mesh that are assumed to have zero space charge density) or contains a single nonzero element which is 1. For locally linear approach each non-zero row has 4 non-zero elements whose magnitudes are dictated by (49).

6.3 APPLICATION OF THE ALGORITHM

With the local charge injection and the \( \rho \)-mesh method described respectively in Sections 6.1 and 6.2 the size of the least squares problem in (48) for the test geometry reduces from 5374 to 65 for the step-wise constant approach and to 87 for the locally linear approach. For such sized least squares problems, the solution time is on the order of seconds and in fact the real bottleneck is the least squares problem itself but the formation of the matrix \( MP_u \) which is prohibitively expensive. The solution of (51) can be used to solve for \( \rho \) with \( \rho = MP_u^{-1}T_u \).

\[
P_{uu} \left[ P_{uu}^{-1}T_u \right] = T_u
\]

Here \( T_u \) and \( U \) are sparse matrices whose multiplication is not numerically expensive. In fact the real bottleneck is the least squares problem itself but the formation of the matrix \( MP_u^{-1} \).

Since explicit formation of the matrix \( P_{uu}^{-1} \) is prohibitively expensive, we find \( P_{uu}^{-1}T_u \) by solving the equation

\[
P_{uu} \left[ P_{uu}^{-1}T_u \right] = T_u
\]

Here \( P_{uu} \) and \( U \) are sparse matrices whose multiplication is not numerically expensive. Equation (51) defines a linear system with multiple right hand sides. Since \( P_{uu} \) is sparse, symmetric and positive-definite the efficient banded Cholesky decomposition method [33] can be used to solve (51) for \( P_{uu}^{-1}T_u \). Since its column number is equal to the size of \( \varrho \), storage of \( P_{uu}^{-1}T_u \) causes no problems. Once \( P_{uu}^{-1}T_u \)
is found $MP_\text{uu}^{-1}T_uU$ is formed by direct multiplication which again is not numerically expensive since $M$ is sparse. Overall the computation time for $MP_\text{uu}^{-1}T_uU$ together with the formation time for matrices $M$, $P_{uu}$ and $T_u$ is on the order of minutes. With $MP_\text{uu}^{-1}T_uU$ found, the least squares problem can be solved by using standard methods. In this paper we use the QR factorization method [33] which is one of the most common methods for solution of a least squares problem. The method QR is named after its main step in which $MP_\text{uu}^{-1}T_uU$ is factorized into an orthogonal matrix denoted by $Q$ and an upper matrix denoted by $R$ such that $MP_\text{uu}^{-1}T_uU = QR$. Since the solution time for the test geometry is on the order of minutes, any reasonable size increase in $\varphi$ and/or $\phi$ in arbitrary geometries is not expected to limit the applicability of FEBKER especially considering that $MP_\text{uu}^{-1}T_uU$ has to be formed only once for each experimental setup.

The least squares problem can be solved to determine $\varphi$. Since the solution time for the test geometry is on the order of minutes, any reasonable size increase in $\varphi$ and/or $\phi$ in arbitrary geometries is not expected to limit the applicability of FEBKER especially considering that $MP_\text{uu}^{-1}T_uU$ has to be formed only once for each experimental setup.

This concludes the initial development of FEBKER. In Figure 12 we show the results of the application to FEBKER to the test geometry. Here we assume that the peak applied ac and dc voltages are the same at 40 kV, the space charge distribution is given in (25) and the points are the reconstruction results. Points A, B, C, D and E are used in Table 1 to illustrate other reconstruction results in Section 7.

![Figure 12. Charge density reconstructions using FEBKER with the locally linear approach as described in the conclusion of Section 6. Here the solid lines are the imposed space charge density of (25) and the points are the reconstruction results. Points A, B, C, D and E are used in Table 1 to illustrate other reconstruction results in Section 7.](image-url)

7. FURTHER IMPROVEMENTS

In (48) the size of the measurement vector $k$ is twice the number of measurements since each measurement can yield two independent parameters $\alpha$ and $\gamma$ as discussed in Section 2. Since the number of unknowns is determined by the size of $\varphi$ in (50), the number of measurements required to make (48) square or overdetermined is at least half the size of $\varphi$. For our test geometry the size of $\varphi$ is 65 for the stepwise constant approach and 87 for the locally linear approach while the number of measurements is 517. The excess number of measurements can be used to improve the applicability of FEBKER which we pursue in this Section.

7.1 ONE PARAMETER MEASUREMENTS

Two parameter Kerr electro-optic measurements require rotating the analyzer during experiments. Rotating the analyzer is slow, making it impractical to investigate time transient charge injection and transport using Kerr electro-optic measurements. The ratio of the number of measurements to the number of unknowns in our test geometry suggests that even with one parameter measurements that do not require analyzer rotation we still can obtain an overdetermined system which can be solved to determine $\varphi$.

To incorporate single parameter measurements into (48), we first normalize (22) to define a normalized intensity measurement $k_n$ and then use (22) and (42) to obtain

$$k_n^{(i)} \equiv \frac{I_n^{(i)}}{I_{nc}} = -\sin 2\theta_a k_n^{(1)} + \cos 2\theta_a k_n^{(2)}$$

where we again use $i$ to index the measurement set. Thus for any fixed analyzer angle, the set of $k_n^{(i)}$ follows from (45) (also see (44)) as

$$[k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(n)}]^T = V k$$

where $n$ is the number of measurements and $V$ is the $n \times 2n$ matrix given by

$$V = [-\sin 2\theta_1, 1, \cos 2\theta_1, 1]$$

with $I$ denoting the $n \times n$ identity matrix. Then for any particular analyzer angle, the one parameter measurement least squares problem is obtained by multiplying (48) with $V$.
\[ V \mathbf{M}_V^{-1} \mathbf{T}_V \mathbf{U}_\Phi - V \mathbf{K} - V (\mathbf{M}_V^{-1} \mathbf{P}_{rad} - \mathbf{M}_d) \Phi_d = 0 \] (55)

where we also substituted (50). Here \( V \mathbf{K} \) is the new measurement vector directly available from intensity measurements. Note that the matrix multiplications introduced by \( V \) do not cause any significant numerical load since \( V \) has only \( 2n \) nonzero elements, where \( n \) is the number of measurements. Thus all the discussion regarding the solution of the least squares problem in Section 6.3 remains valid for (55).

In the fourth and fifth rows of Table 2 we show the data values at points A, B, C, D, and E of Figure 12 when one parameter measurements with \( \theta_a = \pi/2 \) are used. The fourth row uses the locally linear approach and the fifth row uses the step-wise constant approach. The reconstruction results are near perfect and establishes that FEBKER can indeed be used with one parameter measurements. Here we do not show the plot of the full set of reconstruction points since there are no noticeable differences between such plots and the plot in Figure 12; the imposed space charge density and the reconstructed space charge density match near perfectly.

### 7.2 MISSING MEASUREMENTS

Our previous work [8, 28] shows that for point/plane electrode geometries Kerr electro-optic measurements near the needle tip can be extremely noisy to be used as data. For three-dimensional geometries it is also expected that some measurement view angles can be blocked by the electrodes. Here we test FEBKER for such cases by removing measurements that are closer to the needle tip than a distance \( R \) (illustrated in Figure 10) from the measurement set.

The sixth to ninth rows of Table 2 shows our one parameter measurements based reconstruction results at points A, B, C, D, and E of Figure 12 when the measurements near the needle tip are missing. For the sixth and seventh rows the step-wise constant approach is used while for the eighth and ninth rows the locally linear approach is used. For sixth, seventh, eighth and ninth rows the \( R \) of Figure 10 is respectively 0.3, 0.5, 1.0, and 1.3 mm. For other data points in Figure 12 the reconstruction results were near perfect.

The results in sixth and eighth rows show that FEBKER is applicable even when measurements at certain locations are not available. Noise levels on the sixth rows are certainly acceptable while the reconstruction results on the eighth row is identical to the imposed space charge density in the first row. However reconstruction results in seventh and ninth rows for point A illustrate that when there are not enough measurements reconstruction results at certain positions may become undependable. Note that for the locally linear approach the range of measurements that can be left out without any major impact on the reconstructions (1.3 mm) is much larger than the similar range for the step-wise constant approach (0.5 mm).

### 8 IMPACT OF ERROR AND REGULARIZATION

Kerr electro-optic reconstructions are in general ill-posed; even a small amount of noise can have a large effect in the performance of FEBKER. The top reconstruction in Figure 13 illustrates the reconstruction results when 5% random noise is added to synthetic data. Here the reconstruction is based on one-parameter measurements and locally linear approach. The results are typical of ill-posed problems; in the presence of noise a direct application of FEBKER results in completely unintelligible results.

The impact of the noise can be decreased by using some extra information about the space charge distribution and such a methodology is known as regularization. Here as the extra information, we assume that the space charge density distribution is smooth (differentiable). Then one can put a penalty on the variations between neighbor reconstructed values; for reconstructions using the locally linear approach neighbor values are the elements of \( \rho \) that corresponds to neighboring vertices in the \( \rho \)-mesh shown in Figure 8. To illustrate regularization mathematically, we write (55) in the form

\[ A \mathbf{\rho} - \mathbf{b} = 0 \] (56)
\[ A = V M P_{uu}^{-1} T u \phi \]  
\[ b = V k + V [ M P_{uu}^{-1} P_{ud} - M d ] \Phi_{de} \]

The least squares problem in (56) is the minimization of the functional
\[ A(\phi) = ||A\phi - b||^2 \]  
where \( || \) \( \) denotes the two norm of a vector. Instead of (59), the regularized least squares problem minimizes a related functional.
\[ \mathcal{L}(\phi) = A(\phi) + \lambda B(\phi) \]
where \( \lambda \) is a nonnegative constant and \( B(\phi) \) is a measure for variations between neighbor elements of \( \phi \). \( B(\phi) \) is typically expressed in terms of a matrix \( B \)
\[ B(\phi) = ||B\phi||^2 \]

When \( \mathcal{L}(\phi) \) is minimized to find \( \phi \) instead of \( A(\phi) \), the added penalty on the variations suppresses the impact of noise.

Clearly the success of the regularization is dependent on the choice of \( B \) and \( \lambda \). In this paper we do not spend much effort on how to choose the matrix \( B \) and the constant \( \lambda \). In fact this is still an active research topic very much dependent on the application. We leave the investigation of optimal \( B \) and \( \lambda \) for FEBKER to future research. Here we choose \( B \) such that minimization of \( A(\phi) \) is equivalent to minimization of the average difference between the neighbor elements of \( \phi \) and \( \lambda \) is given as
\[ \lambda = \frac{T(A^T A)}{T(B^T B)} \]  
as suggested in [34], with \( T \) indicating the matrix trace.

Bottom reconstructions in Figure 13 show the effects of regularization. The middle reconstruction is based on synthetic data without any added noise. Notice that regularization causes significant distortion at \( r=0.125 \) mm. This distortion is due to our assumed form of space charge density in Figure 5 which is not differentiable at \( r=0 \). Regularization smooths out the space charge density at \( r=0 \) causing lower than expected values. The bottom reconstruction shows the reconstruction when 5% random noise is added to the synthetic data. Remarkably the impact of noise is not very significant which proves that with regularization FEBKER can be used in the presence of noise.

In Figure 13 the distortion caused by regularization at \( r=0.125 \) mm is due to the assumed form of space charge density in (25) since it is not differentiable at the axis of symmetry. Note that this piece-wise linear form is chosen for its simplicity. A comparably simple space charge density distribution which is differentiable at \( r=0 \) is given in (26). The difference between this distribution and the one in (25) is the \( r \) dependence between 0 and 1 mm as illustrated in Figure 5.

In Figure 14 we show the reconstruction results when the space charge distribution in (26) with \( \rho_0=0.12 \) C/m\(^3\) is used instead of the one in (25). As expected the distortion caused by the regularization is considerably less and the impact of noise does not diminish the applicability of FEBKER. These reconstruction results also serve as a proof that the reconstruction results of this paper are not limited to the form of the space charge in (25).

9 CONCLUSIONS

In this paper we developed a new algorithm which is capable of reconstructing the space charge distributions from Kerr electro-optic measurements. The algorithm is built on the FEM for Poisson's equation and is named the FEBKER algorithm. The algorithm is applied to synthetic data generated for a point/plane geometry with and without added noise. The reconstruction results established that the algorithm is ready to be applied to real experimental data which constitutes the next step in our research.

FEBKER has multiple advantages over the onion peeling algorithm which we previously used for Kerr electro-optic measurement of electric field in point/plane electrode geometries. FEBKER reconstructs the space charge distribution instead of the electric field to avoid taking the
divergence which is often very problematic numerically. Standard regularization techniques are directly applicable so that even in the presence of considerable noise in data, FEBKER can reconstruct the space charge accurately. FEBKER is applicable even when each Kerr electro-optic measurement yields one parameter. This makes it easy to use FEBKER for time transient analysis which is difficult with methods based on measurement of two parameters since measurement of two parameters requires rotation of optical elements which is very slow compared to charge transport times. Finally the algorithm is capable of reconstructing the space charge distribution around the needle tip even when Kerr electro-optic measurements very close to the tip are not usable due to extreme noise.

FEBKER is intended for two-dimensional measurements where the light beam is expanded to cover the area of interest and a CCD light detector is used to measure intensity at the output. Two-dimensional measurements are necessary and in fact is the only apparent disadvantage when compared to the onion peeling method which can reconstruct the electric field on a single layer from one-dimensional measurements. Note however that the information gained from single layer reconstructions is very limited and one needs multiple layer reconstructions to have valuable conclusions and that requires two-dimensional measurements anyway. FEBKER can be used for measurements based on non expanded beam (point) measurements. For such measurements it is necessary to move the laser mechanically to get measurements at different positions and thus it is only possible to investigate steady-state or long time constant phenomena.

In this paper we restricted the implementation and application of FEBKER to an axisymmetric point/plane electrode geometry since it is expected that in the near future the algorithm will be applied to mainly point/plane electrode geometries. However the algorithm is directly extendible to arbitrary three-dimensional geometries. The only difficulty is the development of a flexible three-dimensional mesh generator itself or a post processor for commercially/academically available mesh generators so that extensive numerical experimentation becomes practical. This requires extensive effort but is certainly achievable when necessary.

The radius of the curvature of the point electrode used in this work (0.5 mm) is relatively large. For more divergent geometries where the radius of the curvature can be on the order of microns there are potential numerical and experimental challenges. The main numerical challenge is the mesh generation where graded mesh generators tend to be problematic when the length of smaller discretization elements (as small as 1 μm for a very sharp needle) is much smaller than the overall geometry length (10 mm for our example geometry but typically larger for actual experimental setups). The quality of the graded finite element mesh generator is particularly important for such divergent geometries where the point electrode tip must be adequately approximated with very small finite elements but elsewhere the finite element size must be kept large not to exceed solvable finite element problem sizes. From the experimental side our experience shows that for such divergent geometries Kerr electro-optic measurements tend to be very noisy which we attribute to the electrohydrodynamic motion. Also note that the spatial resolution achievable by Kerr electro-optic measurements (expected to be around 0.1 mm) may not be enough to adequately interpret space charge injection for such divergent geometries. For these reasons at this development stage of FEBKER we primarily concentrate on less divergent geometries such as the example geometry in this work and leave numerical and experimental investigation of very divergent geometries to future work.

ACKNOWLEDGMENT

This work has been supported by NSF Grant nos. ECS 9220638 and ECS 9202015.

REFERENCES


Üstündag et al.: Finite Element Based Kerr Electro-Optic Reconstruction of Space Charge


1 Now with Epic Systems Corp., Madison, WI.

Manuscript was received on 12 April 2000, in revised form 4 January 2001.