Unipolar Charge Transport in Oil-Pressboard Systems with Planar, Coaxial Cylindrical and Concentric Spherical Electrode Geometries

*Invited Paper*

Jouya Jadidian and Markus Zahn\*

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

Research Laboratory of Electronics

Laboratory for Electromagnetic and Electronic Systems

High Voltage Research Laboratory

Cambridge, Massachusetts 02139, USA

\*e-mail: zahn@mit.edu

**Abstract**—One-dimensional migration-ohmic unipolar analysis of charge transport phenomena in series, two-region, oil-pressboard composite dielectric systems are presented in planar, coaxial cylindrical and concentric spherical electrode geometries with a step current source. Space charge limited injection solutions are presented in the steady state and transients using the method of characteristics to convert the governing partial differential equations into a set of ordinary differential equations that allow analytical solutions as a function of time and space for volume and interfacial surface charge densities, charge trajectories in the oil region, and electric field and voltage drop in oil and pressboard regions. Analytical solutions are compared with some COMSOL Multiphysics numerical solutions as a way of gaining confidence in the correctness and accuracy of our numerical solutions.

Keywords-mobility; charge transport; interfacial surface charge; method of characteristics

# Introduction

Power transformers, transmission cables, and other high voltage equipment often utilize composite insulation systems. Liquid-solid composite dielectrics, such as transformer oil-pressboard, constitute the major insulation system used to improve the insulation and cooling capabilities of high voltage equipment [1-6]. Experimental evidence has shown that pre-breakdown phenomena in liquid-solid insulation systems often differ from liquid-only systems [1]. Therefore, the ability to model and understand charge injection, transport and interface charge accumulation is of great importance.

A one-dimensional transient analysis of unipolar charge injection and transport in a single dielectric region between two planar electrodes stressed by an applied step voltage was first presented in [7-9]. This work extends the closed-form single region results to two-region series planar, cylindrical, and spherical geometries that model a liquid/solid lossy dielectric system such as transformer oil/pressboard.

The governing partial differential equations in time and space for a step current source are converted into a set of ordinary differential equations in the moving charge’s reference frame using the method of characteristics and solved for space charge limited conditions where the injected charge density from the positive electrode is infinite so that the electric field at the charge injecting electrode is zero keeping the injection current finite. This space-charge limited boundary condition is often assumed in semiconductor applications [10]. In the present work, we have applied this injection boundary condition to two-region series oil/pressboard planar, cylindrical and spherical geometries with a step current source. The oil dielectric is assumed to have unipolar ion conduction described by a constant mobility *μ*, while the pressboard dielectric is modeled by an ohmic conductivity *σ,* with zero volume charge density. The results give charge density and charge trajectories in the oil region, and the electric field distribution as a function of time and space in both oil and pressboard regions. The terminal voltage and surface charge density at the oil/pressboard interface are calculated as a function of time. Most of the analysis yields closed-form expressions which are in good agreement with numerical simulations.

The governing equations in oil and pressboard (pb) regions are:

Oil Pressboard

|  |  |
| --- | --- |
|  | (1) |
|  |  |
|  | (2) |
|  |  |
|  | (3) |

Work supported by ABB Corporate Research, Västerås, Sweden.

In this paper, we first briefly introduce our numerical streamer model in a needle-sphere electrode geometry. The goal of this paper is to give some simple analytical solutions which enable us to check the accuracy of our computer models with one-dimensional planar, cylindrical and spherical electrode geometries. The analytical approach and closed form solutions for planar, cylindrical and spherical electrode geometries are presented in sections III-V. Then we present some simple analytical and numerical solutions from previous sections in Section VI. Section VII describes some numerical results of COMSOL Multiphysics applied to the same one-dimensional electrode geometries solved with closed form solutions in this paper. The analytical and numerical COMSOL results are in excellent agreement being at worst within 0.01% for COMSOL solved spherical geometry transients and at best within 10-10% for COMSOL solved steady-state planar solutions.

# Motivation: Sanity Check of Streamer Models

In this section we briefly introduce our COMSOL Multiphysics streamer model in a needle-plane geometry [3,4]. One of the goals of this paper is to have a sanity check of the accuracy of our numerical model by comparing the results with analytical solutions given in this paper for simple one-dimensional electrode geometries. The governing equations that contain the physics to model streamer development are based on the drift-dominated charge continuity equations (5-7) for positive ion (*ρp*), negative ion (*ρn*) and electron (*ρe*) charge densities which are coupled through Gauss’ law in Eq. (4). The thermal diffusion equation (8) is included to model temperature variations (*T*), heating, and gas formation in oil.

 (4)

 (5)

 (6)

 (7)

 (8)

The negative ion and electron charge densities are both negative quantities. The three carrier continuum model in Eqs. (4-7) is utilized to account for the charge generation and recombination, which are critical in the study of streamers. Since we have assumed that diffusion is negligible in Eqs. (3-7), we have numerically solved charge migration and recombination equations with triangular quartic elements using COMSOL Multiphysics software with three charge carriers.

In Eq. (8), *v*, *kT*, *cv*, and *ρl* are the oil’s velocity, thermal conductivity (0.13 *W*/*m*K), specific heat (1.7×103 J/(kg*-*K)), and mass density (880 kg/m3), respectively, with these values representative for transformer oil [3,4]. In the microsecond time scales of interest for streamer formation the oil’s velocity is negligible such that *v*=0, *q* is the magnitude of electronic charge (1.602×10−19 C) and is the local electric field. The parameters *μp*, *μn*, and *μe* are the mobilities of the positive ions, negative ions and electrons respectively. These parameters are used in the simulations including their dependence on temperature. The values of mobilities and other streamer parameters are given at room temperature in Table I [3,4]. *Rpn* and *Rpe* are the ion-ion and ion-electron recombination coefficients obtained from the Langevin relationship *Rpn* = *Rpe* = *e*(*μp*+*μn*)/ *ε* = 1.64×10−17 m3/s [5]. The two recombination rates are assumed equal because using the Langevin relationship for the ion-electron recombination rate leads to overestimation. The Langevin recombination relationship is a diffusion limited process and valid for situations where the electric field levels are low to moderate and the recombining species are of similar physical scale [3-5]. To compensate for the reduction in the recombination cross-section caused by high electric field levels, some authors have used the Langevin recombination term for ion-ion recombination to model ion/electron recombination [5]. This approach effectively compensates for the reduction in the recombination cross-section by reducing the apparent electron mobility. In addition to recombination, electrons also combine with neutral molecules to form negative ions. This process is modeled as an electron attachment time constant, *τa* = 200 ns.

The generation and recombination terms of the equations play key roles in describing streamer dynamics.  Recombination is described by the Langevin relationship used in Eqs. (5)-(7). The field ionization charge density rate source term, *GF*, was modeled using the Zener model of electron tunneling in solids [6]:

 (9)

Parameter *e*=1.6×10-19 C is the electronic charge,  is the electric field, *h* is Planck’s constant, *m\** is the effective electron mass in the oil (is a function of electric field; for zero electric field, *m\**=0.1×*me* = 9.1×10-32 kg). Other parameter values are listed in Table I. The values used for these parameters are consistent with those used in [3,4]. The main difference between the field ionization function of the present model and older models [3,4] is that we have included the electric field dependence of the ionization potential which has been derived by Density Functional Theory (DFT). Smalø *et al.* have used DFT to find out the effect of the intense electric field on the ionization energy of hydrocarbons [11]. The aromatic oil is basically characterized by the chemical formula of CnHn. The ionization energy for the simplest aromatic hydrocarbon, Benzene C6H6 is derived and fitted by linear regression to a simple equation for ionization potential as listed in Table I as well as other parameters used to solve governing equations (4-9).

TABLE I. Physical parameters used in STREAMER modelS [3,4,11]

|  |  |  |
| --- | --- | --- |
| **Parameter** | **Symbol** | **Value** |
| Number Density | *n*0 | 1×1023 m-3 [3,4] |
| Ionization Potential | *Δ* | eV [11] |
| Molecular Separation | *a* | 3.0 × 10-10 m |
| Ion-ion recombination rate | *Rpn* | 1.64×10-17 m3s-1 |
| Ion-electron recombination rate | *Rpe* | 1.64×10-17 m3s-1 |
| Positive ion mobility | *μp* | 1×10-9 m2V-1s-1 |
| Negative ion mobility | *μn* | 1×10-9 m2V-1s-1 |
| Electron mobility | *μe* | 1×10-4 m2V-1s-1 |

The electric field dependent molecular ionization is the key mechanism for streamer development in transformer oil [3,4]. By ionizing oil molecules into slow positive and negative ions and fast electrons, an area of net space charge quickly develops because the highly mobile electrons are swept away from the ionization zone leaving behind the low mobility positive and negative ions. The net homocharge, with the same polarity as the needle electrode, modifies the electric field distribution in the oil such that the electric field at the needle electrode decreases while the electric field magnitude ahead of the ionized body in the oil increases. The new field distribution leads to ionization occurring further away from the needle electrode, which in turn causes further modification of the electric field and charge distributions. The ultimate result of these electrodynamic processes is the development of ionizing electric field and space charge waves, which are moving dissipative sources that raise the temperature to vaporize transformer oil and create a gas phase. The charge wave has the same polarity as the needle electrode and the oil vaporization leads to the formation of a low density streamer channel in transformer oil. The model predicts inception of bushy negative streamers from a negative needle electrode and thin positive streamers from a positive needle electrode as shown in Figs. 1 and 2. According to COMSOL numerical simulations, negative streamers require higher voltage to initiate in transformer oil such that no highly ionized region forms around the needle electrode unless the applied impulse voltage peak is greater than 400 kV for a 25 µm radius of curvature needle electrode. However, positive streamers initiate at much lower voltages, as can be seen in Fig. 1 for a positively applied 200 kV impulse voltage with 100 ns rise time. It has turned out in our numerical simulations that initial ionized streamer shape and velocity greatly depend on impulse rise-time. Specifically, in positive streamers, slower impulses with the same peak voltage initiate faster and thinner streamer columns, while applied voltage rise-time does not considerably affect the shape and velocity of the initial ionized region around the negative needle. Comparing these results with Fig. 2, which shows results for a negative needle electrode, the analysis indicates that the initially ionized volume is one order of magnitude larger for negative streamers as the high-mobility electrons are repelled by the negative needle electrode and migrate into the liquid bulk resulting in a lower electric field enhancement near the needle electrode, which reduces the ionization rate in the oil. Consequently, negative streamer formation requires a greater impulse voltage peak than their positive counterparts. When a positive impulse voltage is applied, the electrons exit the ionization zone back to the needle electrode which creates a positive space charge region that reduces the electric field near the needle electrode and enhances the electric field ahead of the positive space charge region. The field enhancement makes the positive streamer initiation possible at lower applied voltages. According to the model, for a given applied voltage, negative streamers travel slower than positive streamers. This is again in agreement with experimental records. The results also reveal that the velocity and the shape of positive streamers are approximately insensitive to changes in electron mobility as long as it is about two orders of magnitude greater than the ion mobilities. The same result has been reported in experimental studies [12]. However, the shape and the velocity of negative streamers are quite sensitive to the electron mobility value. Both positive and negative streamers form in different modes in transformer oil depending on the magnitude of applied electric field. Including electric field dependence of ionization potential enables the present model to physically describe the existence of higher modes [3,4] in both positive and negative streamers. Increasing positively applied voltage magnitude leads to greater streamer velocity which can be considered as a 3rd mode for positive streamers [3,4]. The effect of higher voltages can be also seen in negative streamer results. The main difference between higher modes in positive and negative streamers is that the main consequence of increasing the positively applied voltage peak is to increase the streamer velocity and not the electric field ahead of the ionized region while in negative streamers the electric field ahead of the ionized body and the ionization rate is increased as well as causing a velocity ionization wave. The temperature rise in negative streamers is considerably higher than for positive streamers over a larger volume.

Experimental results have shown that negative streamers initiated from the needle electrode generate relatively large gas bubbles in transformer oil [13-15]. These observations support the numerical results we have obtained for negative streamers, since in the larger ionized body around the negative needle, the temperature is about 5000 K, about one order of magnitude higher than the boiling point of transformer oil (~500 *K*). Therefore, it is reasonable to think of a gas phase generated inside the bushy ionized body around the negative needle. On the other hand, the temperature inside the ionized volume of a positive streamer is about or under the boiling point of transformer oil.



Figure1. Two dimensional axisymmetric COMSOL Multiphysics simulations using a 25 µm radius of curvature positive needle electrode for electric field distribution (right side) and net charge density and equipotential surfaces (left side) for positively applied impulse voltage of 200 kV peak and rise time of 100 ns after 100 ns from the start of the impulse voltage. No discharges are observed for a 200 kV negatively applied impulse voltage.



Figure 2. Two dimensional axisymmetric COMSOL Multiphysics simulations for electric field distribution (right side) and net charge density and equipotential surfaces (left side) for negatively applied impulse voltage of 400 kV peak and rise time of 100 ns at time 100 ns after the start of the impulse voltage using a 25 µm radius of curvature needle electrode.

# Planar Geometry

## Governing Equations

A one-dimensional migration-ohmic model, where all quantities only depend on coordinate *x* and time *t*, describes charge transport phenomena in the series two-region oil liquid-pressboard solid planar geometry shown in Fig. 3. Region I represents a transformer oil region with positive charge mobility *μ* and dielectric permittivity *εI= εoil*, while Region II represents a pressboard region with ohmic conductivity *σ* and dielectric permittivity *εII= εpb*. For the transient analysis, we excite the system with a step terminal current *I*(*t*)=*JsAu*(*t*) at *t*=0, where *I*(*t≤* 0-)=0 and *I*(*t* ≥0+)=*JsA* where *Js* is the terminal current per unit electrode area *A* and *u*(*t*) is the unit step function. It is assumed that only the positive electrode at *x=*0 injects positive volume charge into the system. This charge travels from Region I to the interface at *x=a,* where surface charge can accumulate at the interface. The volume charge density in the ohmic Region II will remain zero if the initial volume charge density at *t=0* is zero.



Figure 3: Two-region, series planar, oil-pressboard dielectric model of area *A* excited by a step current source with Region I obeying a mobility (*μ*) conduction law and Region II obeying Ohmic conduction with conductivity *σ*.

In this one-dimensional geometry, the electric field and current density in both regions are only in the *x* direction. Gauss’ law and conservation of charge can be written for the one-dimensional (*x*-direction) oil system in Region I as:

|  |  |
| --- | --- |
|  | (10) |
|  | (11) |

The boundary conditions relating the electric field, current density, and surface charge density at the *x*=*a* interfacial surface are:

 (12)

 (13)

where *σs*(*t*) is the surface charge density at the interface (*x=a*). Since the electric field is conservative (i.e.,) in both regions, the voltage drop between the electrodes is:

|  |  |
| --- | --- |
|  | (14) |

At the positive electrode (*x=*0), the positive charge carriers are injected into Region I via the assumed space charge limited condition where *ρ*(*x=*0*,t*)*=∞* requires that *Eoil*(*x=*0*,t*)*=*0 so that *Joil* remains finite:

|  |  |
| --- | --- |
|  | (15) |

Using Eqs. (10) and (11), the total migration and displacement current densities in Region I are:

|  |  |
| --- | --- |
|  | (16) |

where Js(t)=Is(t)/A is the current source Is per unit electrode area A for t ≥0+. In Region II, the total ohmic and displacement current density in the pressboard is only a function of time and not position:

|  |  |
| --- | --- |
|  | (17) |

## Steady-State Solutions

In the steady-state (*t*→∞), *Joil* (*x*)=*Jpb* (*x*)=*Is*/*S*. Hence, from Eqs. (16) and (17):

|  |  |
| --- | --- |
|  | (18) |

Using Eq. (15), *Eoil*(*x=*0*,t*)*=*0 as a boundary condition at *x*=0, we obtain from (18):

|  |  |
| --- | --- |
|  | (19) |

From Eqs. (14) and (19), the steady state terminal voltage is:

|  |  |
| --- | --- |
|  | (20) |

Equation (20) is rewritten to solve for the current density as a function of voltage:

|  |  |
| --- | --- |
|  | (21) |

From Eq. (19), the steady-state charge density in the oil region (Region I) is found using Gauss’ law:

|  |  |
| --- | --- |
|  | (22) |

The surface charge density *σs* on the interfacial surface *x*=*a* is found from Eqs. (12) and (19):

|  |  |
| --- | --- |
|  | (23) |

For space charge limited injection, the electric field, terminal voltage, current density, charge density and surface charge in Eqs. (19-23), respectively are plotted in section VI.

## Transient Solutions in Region I

Due to the step current source in Fig. 3, the transient analysis results in both migration/ohmic conduction and displacement currents (Eqs. (16) and (17)). We assume an initially unexcited system with space charge limited injection, given by Eq. (15), at *x*=0. From Eqs. (16) and (17), the total current densities in Regions I and II are equal to *Js*(*t*). Also, since *Epb*(*t*) only depends on time and not position, the charge density in Region II is zero, i.e., *ρpb* =*εpb*∂*Epb* /∂*x* =0.

The solution for the electric field in Region I in Figs. (3) and (4) are obtained from the total derivative of the electric field in the reference frame of the moving charge, which from (10), (11), and (16) is:

|  |  |
| --- | --- |
|  | (24) |

Using the method of characteristics [7-9], a general solution method for hyperbolic partial differential equations, the electric field along a charge trajectory is obtained by integrating Eq. (24):

|  |  |
| --- | --- |
|  | (25) |

where *Js*is constant for *t>*0 and *Eoil*(*x=*0*,t*)*=Eoil*(*x,t=*0)=0. Since *dx*/*dt*=*μEoil(x,t)*, the charge trajectory position *x* in Region I is obtained by integrating Eq. (25):

|  |  |
| --- | --- |
|  | (26) |

where 0*≤x*0*≤a* is the starting position of a trajectory at *t*0=0, and *x*0=0 for *t*=*t*0≥0. According to Eqs. (10) and (11), the differential equation which describes the charge density along the trajectories is:

|  |  |
| --- | --- |
|  | (27) |

Using the method of characteristics, Eq. (27) is rewritten as:

|  |  |
| --- | --- |
|  | (28) |

## Initial Condition (Sub-region I1 in Oil):

The general solution of the ordinary differential equation in Eq. (28), for the volume charge density in Region I is:

|  |  |
| --- | --- |
|  | (29) |

where *ρ*(*x*0,*t*0) is the charge density at the starting point of a trajectory at *x=x*0, *t=t*0.For the initial condition in sub-region I1 of Fig. 4, *ρ*(*x*0,*t*0=0)=0. Thus in Sub-region I1 (the area which is labeled as “initial condition problem” in Fig. 4), the charge density is zero, *ρ*(*x*,*t*=0)=0, since it has been assumed that there is no initial volume charge at *t=*0. With zero initial charge density, *ρ*(*x*0*,t*0=0)=0, the charge density *ρ*(*t*) given by Eq. (29) remains zero for its entire trajectory. As a result of (10), with *ρ*(*x,t*)=0, the electric field, *Eoil­* (*t*), for Sub-region I1 is merely a function of time and not position.

## Charge Injection (Sub-region I2 in Oil):

Conversely, after positive charge is injected from the *x=0* positive electrode at *x=0* for *t*0>0, in Sub-region I2, the charge density, *ρ*(*x=*0,*t*=*t*0) for space charge limited injection and the injected charge density is a function of space and time:

|  |  |
| --- | --- |
|  | (30) |

Figure 4 shows the results in the space-time domain for the charge transport transient model. Injected charge in sub-Region I2 travels on specific trajectories (solid black curves) until they reach the interfacial surface at *x*=*a*.



Figure 4: Space-time domain for the transient one-dimensional model of charge transport in the migration-ohmic system for planar electrodes. In Region I, the demarcation curve, *xd(t)* in Eq. (31), separates the initial condition problem (Sub-region I1) from the charge injection problem (Sub-region I2). The integration paths ζ1 and ζ2 in Region I and ξ1 in Region II, used to calculate terminal voltage are shown for times less than the charge time of flight (*td*) starting at *x=*0, *t=*0 and ending at *x=a*, *t=td*. Integration paths ζ3 and ξ2 in Region I /II are shown for times greater than *td*.

The demarcation curve, *xd* (*t*) (Eq. (31)), in Fig. 4, given by Eq. (26) with *x*0=0, *t*0=0, separates Sub-regions I1 and I2. Along this charge trajectory the electric field is given by Eq. (25) with *t*0*=*0, *Eoil*=*Jst/εoil*. To the right of the demarcation curve (Sub-region I2) we have positive charge trajectories emanating from (*x*0*=*0, *t*0>0). Thus, for the demarcation curve *xd* (*t*) and other charge trajectories in Sub-region I1, the electric field is obtained from Eq. (25) with *t*0*=*0. The electric field at *t*=0 is zero, since no charge is yet injected into Region I. The demarcation curve is characterized by Eq. (26) with *x*0*=*0, *t*0*=*0 and *Eoil*(*x*0=0,*t*0=0)=0. The demarcation time, *td*, is the charge time of flight, where *xd* (*t=td*)*=a* with *t*0=0:

|  |  |
| --- | --- |
|  | (31) |

To obtain the electric field in Sub-region I2 at any arbitrary point in the space-time domain, such as (*xi*, *ti*) in Fig. 4, the space charge limited condition, *Eoil*(*x=*0*,t=t*0) =0, serves as an initial condition.

For *t*<*td*, the integration path of electric field for calculation of the voltage drop across Region I should include ζ1 and ζ2 in Sub-regions I2 and I1 (Fig. 4). For *t*>*td*, the integration path ζ3 is entirely in Sub-region I2 (Fig. 4). Thus, the voltage drop across Regions I and II for *t* <*td* can be found by integration paths ζ1 and ζ2 in Region I and ξ1 in Region II; and for *t*>*td* the integration paths are ζ3 in Region I and ξ2 in Region II.

## Transient Solutions in Region II

Solving for the electric field in Region II is more straightforward. From Eq. (17) the electric field in the pressboard region is:

|  |  |
| --- | --- |
|  | (32) |

where we used the initial condition, *Epb*(*t=*0)*=*0*.*

Using Eq. (12) and (13), we can find the interfacial surface charge density as:

|  |  |
| --- | --- |
|  | (33) |

## Transient Voltage Solutions in Regions I and II

By integrating *Eoil* (*t*) in Eq. (25) along ζ1 andζ2 (Fig. 4) for *t* <*td* and along ζ3 for *t* >*td*, we have the voltage drop *Voil*(*t*)across Region I for any time. Also from Eq. (32), *Vpb* (*t*) is found by integrating the electric field *Epb* (*t*) over space in Region II from *x=a* to *x=b*. Adding *Voil* (*t*)and *Vpb* (*t*) from Eq. (14) gives the total voltage drop versus time, which is given by Eq. (34):

|  |  |
| --- | --- |
|  | (34) |

# Cylindrical Geometry

We consider the cylindrical coaxial geometry in Fig. 5, where the inner electrode at *r*=*ri* is a source of ions with mobility *μ* in the oil region (*ri*<*r*<*rm*) and the electric field is purely radial. We assume all physical parameters to be constant and consider only one-dimensional variations with the radial coordinate *r*, so that the current and electric field are only in the radial direction. The general governing equations in oil and pressboard regions are the same as Eqs. (1-3) in cylindrical geometry.



Figure 5: Two-region, series, oil-pressboard dielectric model for cylindrical electrodes excited by a step current source with Region I (Oil) for *ri<r<rm* obeying a mobility (*μ*) conduction law and Region II (Pressboard) for *rm<r<ro* obeying Ohmic conduction.

Again, an infinite amount of charge is available at the charge emitting (*r*=*ri*) electrode for space charge limited injection, so the emitter electric field at *r*=*ri* must be zero to keep the current finite.

## Steady-State Solutions

For steady-state (*t*→∞), the total current *I*(*t*)*=*2*πrDJr*(*r*) is constant. From Eqs. (1)-(3) in cylindrical coordinates, the charge density is

|  |  |
| --- | --- |
|  | (35) |

Using Eqs. (2), (3) and (35):

|  |  |
| --- | --- |
|  | (36) |

By solving Eq. (36), with space charge limited injection, *Eoil* (*ri*)=0, the electric field in the liquid region is:

|  |  |
| --- | --- |
|  | (37) |

Hence, using Eq. (35), the total charge density is then:

|  |  |
| --- | --- |
|  | (38) |

The electric field in the pressboard (Region II) is:

|  |  |
| --- | --- |
|  | (39) |

Integrating the electric-field distribution over *r* in both regions using the integral solution in Region I, *ri<r<rm* :

|  |  |
| --- | --- |
|  | (40) |

yields the steady-state voltage across both regions as:

|  |  |
| --- | --- |
|  | (41) |

## Transient Solutions in Region I

Due to the step current source in Fig. 5, the transient analysis results in both migration/ohmic conduction and displacement currents.

We again assume an initially unexcited system with space charge limited conditions at *r*=*ri*. Hence, the radial current densities in Regions I and II are equal to *Js*(*t*). Also, since the charge density in the pressboard region is zero, the electric field in the pressboard region, *Epb(t)*, depends on time and position as 1/*r*:

|  |  |
| --- | --- |
|  | (42) |

To solve for the time dependent charge density in the oil region, based on Eqs. (1-3), we have:

|  |  |
| --- | --- |
|  | (43) |

which can be reduced to:

|  |  |
| --- | --- |
|  | (44) |

According to the method of characteristics, Eq. (44) can be rewritten as a pair of ordinary differential equations:

|  |  |
| --- | --- |
|  | (45) |

The cylindrical differential equation for charge density is identical to the analogous planar geometry equation in Eq. (27) and so has solutions from Eqs. (28) and (29) and Fig. 6:

|  |  |
| --- | --- |
| (46) | (46) |

Again, from Eqs. (2), (35) and (42), we have an ordinary differential equation for electric field in the oil regions I1 and I2 for which using the method of characteristics gives:

|  |  |
| --- | --- |
|  | (47) |

with charge trajectories:

|  |  |
| --- | --- |
|  | (48) |

Note again that the demarcation curve, *rd* (*t*) is the charge trajectory that separates the initial condition problem from the charge injection problem which starts from *r=ri* at *t*0=0 in Eq. (48). By solving Eq. (48) we obtain similar regions and sub-regions as the planar geometry in Fig. 4, but now for cylindrical geometry as shown in Fig. 6. The demarcation curve *rd(t)* and time of flight *td* for cylindrical electrodes shown in Fig. 6 is:

|  |  |
| --- | --- |
|  | (49) |

The electric field in sub-region I1 above the demarcation curve, where *ρ*=0 is:

|  |  |
| --- | --- |
|  | (50) |

where *rs* is the starting point of a charge trajectory between *ri* and *rm* at *t=0*. In sub-region I2, where *ρ*=*εoil*/[*μ*(*t*-*t*0)] we have:

|  |  |
| --- | --- |
|  | (51) |

where the trajectories start at *r=r*i and *t=t*0.



Figure 6: Space-time domain for the transient one-dimensional model of charge transport in the migration-ohmic system for coaxial cylindrical electrodes. In Region I, the demarcation curve, *rd(t),* separates the initial condition problem (Sub-region I1) from the charge injection problem (Sub-region I2). The integration paths ζ1 and ζ2 in Region I and ξ1 in Region II, used to calculate terminal voltage are shown for times less than the charge time of flight (*td*) starting at *r=ri*, *t=*0 and ending at *r=rm*, *t=td*. Integration paths ζ3 and ξ2 in Region I /II are shown for times greater than *td*.

## Transient Solutions in Region II

Solving for the electric field in Region II is more straightforward. By solving the differential equation of the last term in Eq. (42) in cylindrical geometry (analogous to Eq. (17) for planar geometry), the electric field in the pressboard region is:

|  |  |
| --- | --- |
|  | (52) |

To find the interfacial surface charge density we use the boundary condition at *r=rm*:

|  |  |
| --- | --- |
|  | (53) |

which yields:

|  |  |
| --- | --- |
|  | (54) |

where *td* is given in Eq. (49).

Again, integrating the electric-field distributions over *r* in both regions using the identity of Eq. (40) yields the transient voltage across regions as Eq. (55):

|  |  |
| --- | --- |
|  | (55) |

where the demarcation curve is given in Eq. (49).

# Spherical Geometry

Throughout this paper, a lower case roman r is used for spherical radial coordinates while an italicized *r* is used for cylindrical radial coordinates. We consider the spherical geometry in Fig. 7, where the inner electrode at r=r*i* is a source of ions with mobility *μ* in the oil region (r*i*<r<r*m*) and the electric field is purely radial. We assume all physical parameters to be constant and consider only one-dimensional variations with the radial coordinate *r*, so that the current and electric field are only in the radial direction. The general equations are assumed to be the same as Eqs. (1-3) in spherical geometry. Again, an infinite amount of charge is available at the charge emitting (r=r*i*) electrode, so that the emitter electric field at r=r*i* must be zero to keep the current finite.



Figure 7: Two-region, series, oil-pressboard dielectric model for spherical electrodes excited by a step current source with Region I (Oil) ri<r<rm obeying a mobility (*μ*) conduction law and Region II (Pressboard) rm<r<r0 obeying Ohmic conduction.

## Steady-State Solutions

In the steady-state (*t*→∞), the total current *I*(*t*)*=*4*πr2Jr*(*r*) is constant. From Eqs. (1)-(3) in spherical coordinates, the charge density is only in the oil region (Region I in Fig. 8):

|  |  |
| --- | --- |
|  | (56) |

Using Eqs. (2) ,(3) and (56):

|  |  |
| --- | --- |
| (57) | 99 |

By solving Eq.(57), with space charge limited injection, *Eoil* (r*i*)=0, the steady state electric field in the oil region is:

|  |  |
| --- | --- |
|  | (58) |

Hence, using Eq. (57), the steady state charge density is then:

|  |  |
| --- | --- |
|  | (59) |

and the electric field in the pressboard region (Region II) in Fig. 8 is:

|  |  |
| --- | --- |
|  | (60) |

## Transient Solutions in Region I

Due to the step current source in Fig. 7, the transient analysis results in both migration/ohmic conduction and displacement currents:

|  |  |
| --- | --- |
|  | (61) |

We again assume an initially unexcited system with space charge limited injection at r = r*i*. Hence, the current densities in Regions I and II are equal to *Js*(*t*). Also, since the electric field in the pressboard region, *Epb*(r,*t*), only depends on time and radial position as 1/r2, the charge density in Region II is zero. To solve for the charge density in the oil region (Region I) , based on Eqs. (1-3) in spherical coordinates, we have:

|  |  |
| --- | --- |
| (62) |  |

which reduces to:

|  |  |
| --- | --- |
|  | (63) |

According to the method of characteristics, Eq. (63) can be rewritten as a pair of ordinary differential equations:

|  |  |
| --- | --- |
| (64) |  |

Considering boundary conditions of zero volume charge at *t=0* and space charge limited charge injection at *r=ri*, the distribution of charge density is of identical form as planar and cylindrical geometries given by Eqs. (29), (30) and (46):

|  |  |
| --- | --- |
|  | (65) |

Again, from Eqs. (2), (56) and (61), we find an ordinary differential equation in terms of electric field in the oil region using the method of characteristics:

|  |  |
| --- | --- |
|  | (66) |

The electric field and charge trajectories are then:

|  |  |
| --- | --- |
|  | (67) |

|  |  |
| --- | --- |
|  | (68) |

The demarcation curve, *rd(t)* and time of flight *td* for spherical electrodes shown in Fig. 8 are:

|  |  |
| --- | --- |
|  | (69) |

|  |  |
| --- | --- |
|  | (70) |

The demarcation curve as shown in Fig. 8 for spherical geometry is the charge trajectory that separates the initial condition problem from the charge injection problem that starts from *r=ri* at *t*=0 in Eq. (67). By solving Eqs. (66)-(68) we have similar regions and sub-regions as in Figs. 4 and 6 as shown in Fig. 8 for spherical geometry. Above the demarcation curve (I1), where *ρ*=0 at *t*0=0 we have:

|  |  |
| --- | --- |
|  | (71) |

where rs is the starting position with r*i*<r<r*m* at *t*0=0. In sub-region I2, where *ρ*=*εoil*/[*μ*(*t*-*t*0)] we have:

|  |  |
| --- | --- |
|  | (72) |

## Transient Solutions in Region II

By solving Eq. (61) in spherical geometry (analogous to Eq. (17) for planar geometry and Eq. (42) for cylindrical geometry), the electric field in the pressboard region is:

|  |  |
| --- | --- |
|  | (73) |

To find the interfacial surface charge density at r=r*m* we have:

|  |  |
| --- | --- |
|  | (74) |

which yields:

|  |  |
| --- | --- |
|  | (75) |



Figure 8: Space-time domain for the transient one-dimensional model of charge transport in the migration-ohmic system for concentric spherical electrodes. In Region I, the demarcation curve separates the initial condition problem (Sub-region I1) from the charge injection problem (Sub-region I2). The integration paths ζ1 and ζ2 in Region I and ξ1 in Region II, used to calculate terminal voltage are shown for times less than the charge time of flight (*td*) starting at r*=*ri, *t=*0 and ending atr*=*rm, *t=td*. Integration paths to calculate terminal voltage ζ3 and ξ2 in Region I /II are shown for times greater than *td*.

To find the voltage across Region I, we have to evaluate the integral:

|  |  |
| --- | --- |
|  | (76) |

which we evaluate by numerical methods in Section VI.

# Results and Discussion

In this section we present plots of some analytical solutions from the previous sections as well as some numerical results in spherical geometry. In the model, we have selected representative values in Table II for transformer oil and pressboard.

Figure 9 shows the transient voltages of planar and cylindrical geometries which have been obtained analytically in Eqs. (34) and (55). The voltage for spherical geometry was obtained using numerical integration methods to evaluate the integral of Eq. (76).

Table II. Numerical Parameter Values of dielectric model

|  |  |  |
| --- | --- | --- |
| **Parameter** | **Symbol** | **Value** |
| Permittivity of oil region | *εoil* | 2**×**10-11 Fm-1 |
| Permittivity pressboard region | *εpb* | 4**×**10-11 Fm-1 |
| Positive ion mobility in oil | *μ* | 10-9 m2V-1s-1 |
| Conductivity of pressboard region | *σ* | 3**×**10-12 Ω-1m-1 |
| Planar geometry interfaces | *a*, *b* | 0.0125 m, 0.025 m |
| Cylindrical geometry interfaces | *ri*, *rm*, *ro*, *D* | 0.0125 m, 0.025 m, 0.0375 m, 0.05 m |
| Spherical geometry interfaces | r*i*,r*m*, r*o* | 0.0125 m, 0.025 m, 0.0375 m |
| Current density at positive electrode | *Js* | 2.5**×**10-7 Am-2 |



Figure 9: Non-dimensionalized voltage between electrodes for the three electrode geometries treated in this paper as a function of non-dimensionalized time ****where all parameter values are defined in Table II, and in particular *a=*0.0125 m is the oil region thickness in planar, cylindrical and spherical geometries.

Figure 10 shows the steady state electric field versus position in both oil and press-board regions of planar, cylindrical and spherical geometries. This graph is shown with a logarithmic scale on the vertical axis. The jump in electric field at the interface between oil and pressboard materials at =0.5 is due to both the discontinuity in dielectric constants and due to the steady state surface charge distribution.

Figure 11 shows the demarcation curves in the space-time domain in the oil region which separates the initial condition problem (Sub-region I1 in Figs. 4, 6 and 8) from the charge injection problem (Sub-region I2 in Figs. 4, 6 and 8) for planar, cylindrical and spherical geometries.

Figure 12 shows the steady state volume charge density versus position in the oil regions of planar, cylindrical and spherical geometries.



Figure 10: Non-dimensionalized steady state electric field between electrodes for different electrode geometries. Non-dimensionalized distance between positive electrode and interfacial surface is defined as  which is equal to *x/*(2*a*) for planar geometry, *r/*(2*a*)-0.5 for cylindrical geometry and r*/*(2*a*)-0.5 for spherical geometry. All parameter values are defined in Table II, and in particular *ri=*r*i=a=*0.0125 m is the radius of the interior electrode in cylindrical and spherical geometries respectively.



Figure 11: Non-dimensionalized space-time, trajectories for different electrode geometries which show the demarcation curves in planar, cylindrical and spherical geometries. Variable *S* is the demarcation trajectory *x, r,* and r given in Eqs. (31), (49), and (69) for planar, cylindrical, and spherical geometries, respectively. Non-dimensionalized distance between the positive electrode and the interfacial surface is defined as  which is equal to *x/*(2*a*) for planar geometry, *r/*(2*a*)-0.5 for cylindrical geometry and r*/*(2*a*)-0.5 for spherical geometry as a function of non-dimensionalized time **** where all parameter values are defined in Table II, and in particular *ri=*r*i=a=*0.0125 m is the radius of the interior electrode in cylindrical and spherical geometries, respectively.



Figure 12: Non-dimensionalized volume charge density in the oil regions between electrodes for different electrode geometries in steady state. Non-dimensionalized distance between the positive electrode and interfacial surface is defined as  which is equal to *x/*(2*a*) for planar geometry, *r/*(2*a*)-0.5 for cylindrical geometry and r*/*(2*a*)-0.5 for spherical geometry.

Figure 13 shows the transient interfacial surface charge density of planar, cylindrical and spherical geometries which have been obtained analytically in Eqs. (33), (54) and (75) respectively.



Figure 13: Nondimensionalized electric surface charge density at the oil/pressboard interface for planar, cylindrical and spherical geometries as a function of non-dimensionalized time ****.

# Numerical Simulations

We have also performed some numerical simulations using COMSOL Multiphysics to confirm analytical solutions we obtained in this paper for planar, cylindrical and spherical electrode geometries. The model assumes charge migration in the oil region and ohmic conduction in the pressboard region. We have applied the COMSOL electro-quasi-static module with convection and diffusion modules of COMSOL for solving charge migration conduction in oil using Eqs. (1,2) and the first part of Eq. (3). For the ohmic pressboard region we have used the in-plane electric currents module of COMSOL to solve Eq. (1) and the second (pressboard) part of Eq. (3). Proper boundary conditions have been employed to exactly match with those assumptions we have taken in our analyses. The numerical results closely match with analytical solutions with error less than 0.01 percent.

# References

1. L. Lundgaard, D. Linhjell, G. Berg, and S. Sigmond, “Propagation of positive and negative streamers in oil with and without pressboard interfaces,” *IEEE Trans. Dielectr. Electr. Insul.*, vol. 5, no. 3, pp. 388-395, June 1998.
2. J. Jadidian, J. G. Hwang, M. Zahn, N. Lavesson, O. Widlund, K. Borg, "Migration-Ohmic Charge Transport in Liquid-Solid Insulation Systems," presented at International Conference on Dielectric Liquids, Trondheim, Norway, 2011.
3. J. G. Hwang, “Elucidating the mechanisms behind prebreakdown phen-omena in transformer oil systems,” *Ph.D. dissertation, Massachusetts Institute of Technology*, Cambridge, MA, USA, June 2010.
4. J. G. Hwang, M. Zahn, L. A. A. Pettersson, O. Hjortstam, and R. Liu, “Modeling streamers in transformer oil: The transitional fast 3rd mode streamer, ” In Proc. IEEE International Conference on the Properties and Applications of Dielectric Materials ICPADM, pp. 573-578, 2009.
5. W. F. Schmidt. Liquid State Electronics of Insulating Liquids. CRC Press, 1997.
6. C. Zener. A theory of the electrical breakdown of solid dielectrics. Proc. Roy. Soc. A, pp. 523-529, 1934.
7. M. Zahn, C. F. Tsang and S. C. Pao, “Transient electric field and space-charge behavior for unipolar ion conduction,” *J. Appl. Phys.*, vol. 45, pp. 2432 -2440, 1974.
8. M. Zahn, “Transient Drift-Dominated Unipolar Conduction Between Concentric Cylinders and Spheres, “IEEE Trans. Elect. Insulation, vol. EI-11, no. 4, pp. 150-157, Dec 1076.
9. M. Zahn, H. Chatelon, “Charge injection between concentric cylindrical electrodes, “Journ. Applied Phys., vol. 45, no. 5, pp. 1797-1805, May 1977.
10. M. A. Lampert and P. Mark, *Current Injection In Solids*, Academic, New York, 1970.
11. H. S. Smalø, Ø. Hestad, S. Ingebrigtsen, and P. O. Åstrand, “Field dependence on the molecular ionization potential and excitation energies compared to conductivity models for insulation materials at high electric fields, “*J. Applied Phys.*, 109, 109, 073306, 2011.
12. J. C. Devins, S. J. Rzad, R. J. Schwabe, “Breakdown and prebreakdown phenomena in liquids, “*J. Appl. Phys*. 52(7), 1981.
13. K. C. Kao, J. P. C. McMath, “Time dependent pressure effect in liquid dielectrics,” *IEEE Trans. Elec. Insulation*, vol. EI-5, no. 3, pp. 64-68, 1970.
14. M. Cevallos, M. Butcher, J. Dickens, A. Neuber, H. Krompholz, “Imaging of Negative Polarity DC Breakdown Streamer Expansion in Transformer Oil due to Variations in Background Presssure”, *IEEE Trans on Plasma Sci.*, vol. 33, pp 494-495, 2005.
15. O. Lesaint, R. Tobazeon, “Streamer Generation and Propagation in Transformer Oil under ac Divergent Field Conditions, “*IEEE Trans. Electr. Ins.*, vol. 23, no. 6 pp. 941-954, 1988.