Photon-Atom Interactions

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ACADEMIC PRESS, INC.

Harcourt Brace Jovanovich, Publishers

Boston San Diego New York
Berkeley London Sydney
Tokyo Toronto

or

$$a_H(t) = a_H(0)e^{-i\omega t} = ae^{-i\omega t},$$
$$a_H^\dagger(t) = a_H^\dagger(0)e^{i\omega t} = a^\dagger e^{i\omega t}. \tag{4.83}$$

These equations also be obtained on the basis of the following theorem [3]. Let

$$F(x) = e^{xa^\dagger a}ae^{-xa^\dagger a} \tag{4.84a}$$

$$\frac{dF}{dx} = e^{xa^\dagger a}(a^\dagger aa - aa^\dagger a)e^{-xa^\dagger a}. \tag{4.84b}$$

With the help of the commutation rule (Eq. (4.71)),

$$\frac{dF}{dx} = -e^{xa^\dagger a}ae^{-xa^\dagger a} = -F(x) \tag{4.85}$$

so that

$$F(x) = F(0)e^{-x} \tag{4.86a}$$

or

$$e^{xa^\dagger a}ae^{-xa^\dagger a} = ae^{-x}. \tag{4.86b}$$

We now may apply this theorem to transform a and $a^\dagger$ to the Heisenberg representation. According to Eq. (2.105), with $t_0 = 0$,

$$a_H(t) = e^{i\mathscr{H}t/\hbar}ae^{-i\mathscr{H}t/\hbar}. \tag{4.87}$$

Substituting Eq. (4.72) for the Hamiltonian and referring to Eq. (4.86) with $x = i\omega t$,

$$a_H(t) = e^{i\omega t a^\dagger a}ae^{-i\omega t a^\dagger a} = ae^{-i\omega t}. \tag{4.88}$$

The conversion of $a^\dagger$ follows in the same fashion.

9. In the interaction representation, with $\mathscr{H} = \mathscr{H}_0 + V$ and $\mathscr{H}_0 = \hbar\omega(a^\dagger a + 1/2)$,

$$\tilde{a}(t) = ae^{-i\omega t}, \qquad \tilde{a}^\dagger(t) = a^\dagger e^{i\omega t}. \tag{4.89}$$

10. There are several definitions of boson operator products. Consider, for example, the product $A = a^\dagger aa^\dagger$. By use of the commutation rule $[a, a^\dagger] = 1$, we may write A in either of the two forms

$$A_n = a^\dagger a^\dagger a + a^\dagger, \qquad A_a = aa^\dagger a^\dagger - a^\dagger. \tag{4.90}$$

In A_n all creation operators are placed to the left of annihilation operators, while in A_a the operators are reversed in position—creation operators to the right of annihilation operators. A_n is said to be in *normal order* and A_a in

antinormal order. Although the operator products are written in different forms, they are nevertheless equal, i.e.,

$$A = A_n = A_a. \tag{4.91}$$

Operators in normal order, such as A_n, are often written $:A:$. To generalize these statements, any function $f(a, a^\dagger)$ that can be expressed as a power series in a and $a^\dagger$ may be converted by means of the commutation rule into normal order

$$:f(a, a^\dagger): \equiv f_n(a, a^\dagger) = \sum_{ij} \alpha_{ij}(a^\dagger)^i a^j \tag{492a}$$

or antinormal order

$$f_a(a, a^\dagger) = \sum_{ij} \beta_{ij} a^i (a^\dagger)^j, \tag{4.92b}$$

where α_{ij} and β_{ij} are numerical coefficients. Thus, if

$$f(a, a^\dagger) = (a + a^\dagger)^2 = a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}, \tag{4.93a}$$

the normal order is

$$:f(a, a^\dagger): = a^2 + 2a^\dagger a + a^{\dagger 2} + 1, \tag{4.93b}$$

and the antinormal order is

$$f_a(a, a^\dagger) = a^2 + 2aa^\dagger + a^{\dagger 2} - 1. \tag{4.93c}$$

4.4 Quantized Fields, Photon-Number States

Now that we have the quantized form of the harmonic oscillator as described in the previous section, the process of quantizing the radiation field [1–4] becomes a matter of simply reinterpreting the canonical variables $Q_{\mathbf{k}\lambda}$ and $P_{\mathbf{k}\lambda}$ in Eq. (4.62) as operators that satisfy commutation relations similar to Eq. (4.68). We therefore postulate

$$[Q_{\mathbf{k}\lambda}, P_{\mathbf{k}'\lambda'}] = i\hbar\,\delta_{\mathbf{k}\mathbf{k}'}\,\delta_{\lambda\lambda'},$$
$$[Q_{\mathbf{k}\lambda}, Q_{\mathbf{k}'\lambda'}] = [P_{\mathbf{k}\lambda}, P_{\mathbf{k}'\lambda'}] = 0. \tag{4.94}$$

The annihilation and creation operators, defined by

$$a_{\mathbf{k}\lambda} = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k Q_{\mathbf{k}\lambda} + iP_{\mathbf{k}\lambda}), \qquad a_{\mathbf{k}\lambda}^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k Q_{\mathbf{k}\lambda} - iP_{\mathbf{k}\lambda}), \tag{4.95a}$$

$$Q_{\mathbf{k}\lambda} = \sqrt{\frac{\hbar}{2\omega_k}}(a_{\mathbf{k}\lambda}^\dagger + a_{\mathbf{k}\lambda}), \qquad P_{\mathbf{k}\lambda} = i\sqrt{\frac{\hbar\omega_k}{2}}(a_{\mathbf{k}\lambda}^\dagger - a_{\mathbf{k}\lambda}), \tag{4.95b}$$

then satisfy the boson commutation rules

$$[a_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'},$$
$$[a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}] = [a^\dagger_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}] = 0. \tag{4.96}$$

As in the case of the harmonic oscillator, the Hamiltonian for the radiation field can be written in terms of the annihilation and creation operators

$$\mathscr{H} = \sum_{\mathbf{k}\lambda} \mathscr{H}_{\mathbf{k}\lambda} = \sum_{\mathbf{k}\lambda} \hbar\omega_k\left(a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} + \frac{1}{2}\right) = \sum_{\mathbf{k}\lambda} \hbar\omega_k\left(N_{\mathbf{k}\lambda} + \frac{1}{2}\right) \tag{4.97}$$

in which

$$N_{\mathbf{k}\lambda} = a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} \tag{4.98}$$

is the *photon number* or *occupation number* operator for the $\mathbf{k}\lambda$ mode of the radiation field. The eigenvalue equations

$$N_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = n_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle, \tag{4.99a}$$
$$\mathscr{H}_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = E_{k\lambda}|n_{\mathbf{k}\lambda}\rangle = \hbar\omega_k(n_{\mathbf{k}\lambda} + \tfrac{1}{2}), \tag{4.99b}$$

with

$$\langle n_{k\lambda}|n_{k'\lambda'}\rangle = \delta_{kk'}\delta_{\lambda\lambda'} \tag{4.100}$$

define the *photon-number* states $|n_{k\lambda}\rangle$ whose eigenvalues $n_{k\lambda}$ are positive integers, 0, 1, 2, ... By analogy with Eq. (4.98), we have

$$a_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}}\,|n_{\mathbf{k}\lambda} - 1\rangle, \qquad a_{\mathbf{k}\lambda}|0\rangle = 0,$$
$$a^\dagger_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda} + 1}\,|n_{\mathbf{k}\lambda} + 1\rangle. \tag{4.101}$$

Also, as in Eq. (4.80b),

$$[a_{\mathbf{k}\lambda}, \mathscr{H}] = \hbar\omega_k a_{\mathbf{k}\lambda}, \qquad [a^\dagger_{\mathbf{k}\lambda}, \mathscr{H}] = -\hbar\omega_k a^\dagger_{\mathbf{k}\lambda}. \tag{4.102}$$

The transition to quantum mechanics by means of the preceding formalism lends itself to the following interpretation: $a_{\mathbf{k}\lambda}$ and $a^\dagger_{\mathbf{k}\lambda}$ are annihilation and creation operators, respectively, for a *photon* with propagation vector $\mathbf{k}$, polarization $\hat{\varepsilon}_{\mathbf{k}\lambda}$, momentum $\hbar\mathbf{k}$, frequency ω_k, and energy $\hbar\omega_k$. The frequency ω_k depends on the magnitude k but is independent of the direction of propagation and the polarization. In the quantized field, $n_{k\lambda}$ gives the number of photons characterized by $\mathbf{k}$ and λ.

It should be recognized that a pure photon-number state is an abstraction not unlike that of an ideal classical monochromatic electromagnetic wave. Neither can be realized in practice; nevertheless, the concepts are very useful and help us to understand the behavior of real systems.

The operators $a_{\mathbf{k}\lambda}$ and $a^\dagger_{\mathbf{k}\lambda}$ are in the Schrödinger representation. To convert them to the Heisenberg and interaction representations, one simply may follow the pattern established by Eqs. (4.83) and (4.89).

$$\{a_{\mathbf{k}\lambda}(t)\}_{\mathrm{H}} = a_{\mathbf{k}\lambda}e^{-i\omega_k t}, \qquad \{a^\dagger_{\mathbf{k}\lambda}(t)\}_{\mathrm{H}} = a^\dagger_{\mathbf{k}\lambda}e^{i\omega_k t}. \tag{4.103}$$
$$\tilde{a}_{\mathbf{k}\lambda}(t) = a_{\mathbf{k}\lambda}e^{-i\omega_k t}, \qquad \tilde{a}^\dagger_{\mathbf{k}\lambda}(t) = a^\dagger_{\mathbf{k}\lambda}e^{i\omega_k t}. \tag{4.104}$$

A complete specification of the quantized radiation field consists of an enumeration of the photon numbers $n_{\mathbf{k}\lambda}$. Since each mode is independent, and the total Hamiltonian $\mathscr{H}$ is the sum of all the partial Hamiltonians $\mathscr{H}_{\mathbf{k}\lambda}$, the product of all the eigenstates $|n_{\mathbf{k}\lambda}\rangle$ is an eigenstate of $\mathscr{H}$. A multimode photon-number state of the field therefore is written

$$|n_{\mathbf{k}_1\lambda_1}\rangle|n_{\mathbf{k}_2\lambda_2}\rangle\cdots|n_{\mathbf{k}_j\lambda_j}\rangle\cdots \equiv |n_{\mathbf{k}_1\lambda_1}, n_{\mathbf{k}_2\lambda_2}, \ldots, n_{\mathbf{k}_j\lambda_j}, \ldots\rangle. \tag{4.105}$$

In a less cumbersome notation, with

$$n_{\mathbf{k}_j\lambda_j} \equiv n_j, \tag{4.106}$$

the multimode photon-number state is written

$$|n_1, n_2, \ldots, n_j, \ldots\rangle = |\{n_i\}\rangle = \prod_i |n_i\rangle. \tag{4.107}$$

A state of this type is an eigenstate of the field Hamiltonian (Eq. (4.97))

$$\mathscr{H}|\{n_i\}\rangle = E|\{n_i\}\rangle, \qquad E = \sum_i \hbar\omega_i\left(n_i + \frac{1}{2}\right), \tag{4.108}$$

and is subject to the orthogonality condition

$$\langle n_1, n_2, \ldots, n_j, \ldots|n'_1, n'_2, \ldots, n'_j, \ldots\rangle = \delta_{n_1n'_1}\delta_{n_2n'_2}\cdots\delta_{n_jn'_j}\cdots. \tag{4.109}$$

With the definitions

$$a_j \equiv a_{\mathbf{k}_j\lambda'_j} \qquad a^\dagger_j \equiv a^\dagger_{\mathbf{k}_j\lambda'_j} \tag{4.110}$$

the relations analogous to Eq. (4.78a) are

$$a_j|n_1, n_2, \ldots, n_j, \ldots\rangle = n_j^{1/2}|n_1, n_2, \ldots, n_j - 1, \ldots\rangle,$$
$$a_j|n_1, n_2, \ldots, 0_j, \ldots\rangle = 0 \tag{4.111}$$
$$a^\dagger_j|n_1, n_2, \ldots, n_j, \ldots\rangle = (n_j + 1)^{1/2}|n_1, n_2, \ldots, n_j + 1, \ldots\rangle.$$

When a particular $n_{\mathbf{k}\lambda}$ is zero, the mode $\mathbf{k}\lambda$ is in its lowest state and when $n_{\mathbf{k}\lambda} = 0$ for all $\mathbf{k}$, the field as a whole is said to be in the ground or *vacuum* state, designated by

$$|0\rangle \equiv |\mathrm{vac}\rangle \equiv |0_1, 0_2, \ldots, 0_j, \ldots\rangle. \tag{4.112}$$

For this state,

$$a_j|0\rangle = 0 \tag{4.113}$$

for all j, and by repeated use of the creation operator, it is possible to generate an arbitrary state of the field,

$$|\{n_i\}\rangle = \prod_i \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}}|0\rangle. \tag{4.114}$$

In the course of the transition from classical to quantum mechanics, embodied in the commutation rules (Eq. (4.94) or (4.96)), the classical vector potential (Eq. (4.44)) becomes a quantum mechanical operator. This is a result of both the connection between $A_{\mathbf{k}\lambda}$, $A^*_{\mathbf{k}\lambda}$ and $Q_{\mathbf{k}\lambda}$, $P_{\mathbf{k}\lambda}$, as in Eq. (4.58) and the relation of the latter quantities to the annihilation and creation operators $a_{\mathbf{k}\lambda}$, $a^\dagger_{\mathbf{k}\lambda}$ given by Eq. (4.95). We now may express the classical $\rightarrow$ quantum mechanical transition as follows:

$$\begin{aligned} A_{\mathbf{k}\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} &\rightarrow \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\, a_{\mathbf{k}\lambda}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \\ &= \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\,(a_{\mathbf{k}\lambda})_{\mathrm{H}}e^{i\mathbf{k}\cdot\mathbf{r}}, \end{aligned} \tag{4.115}$$

where the subscript H denotes that the operator is in the Heisenberg representation. A similar expression is written for the complex conjugate term; hence, for the complete vector potential, the transition from the classical to the quantum mechanical form is given by

$$\mathbf{A}(\mathbf{r},t) \rightarrow \mathbf{A}_{\mathrm{H}}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\,\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}\lambda}[(a_{\mathbf{k}\lambda})_{\mathrm{H}}e^{i\mathbf{k}\cdot\mathbf{r}} + (a^\dagger_{\mathbf{k}\lambda})_{\mathrm{H}}e^{-i\mathbf{k}\cdot\mathbf{r}}]. \tag{4.116}$$

Both sides of the quantum mechanical vector potential now may be transformed to the Schrödinger representation, giving

$$\mathbf{A}(r) = \sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\,\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}\lambda}[a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} + a^\dagger_{\mathbf{k}\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}]. \tag{4.117}$$

The quantized forms of $\mathbf{E}$ and $\mathbf{B}$ follow directly from the basic relations $\mathbf{E} = -\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$. To obtain the electric field $\mathbf{E}$, we revert to the Heisenberg representation and the corresponding equation of motion:

$$\mathbf{E}_{\mathrm{H}}(\mathbf{r},t) = -\frac{\partial\mathbf{A}_{\mathrm{H}}(\mathbf{r},t)}{\partial t} = \frac{i}{\hbar}[\mathbf{A}_{\mathrm{H}}(\mathbf{r},t),\mathscr{H}]. \tag{4.118}$$

With $\mathbf{A}_{\mathrm{H}}(\mathbf{r},t)$ provided by Eq. (4.116), the commutator is evaluated by referring to Eq. (4.82). The result is

$$\mathbf{E}_{\mathrm{H}}(\mathbf{r},t) = i\sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}\lambda}[(a_{\mathbf{k}\lambda})_{\mathrm{H}}e^{i\mathbf{k}\cdot\mathbf{r}} - (a^\dagger_{\mathbf{k}\lambda})_{\mathrm{H}}e^{-i\mathbf{k}\cdot\mathbf{r}}], \tag{4.119a}$$

and the corresponding electric field in the Schrödinger representation is

$$\mathbf{E}(\mathbf{r}) = i\sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}\lambda}[a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}]. \tag{4.119b}$$

The $\mathbf{B}$ field is derived directly from Eq. (4.117):

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \nabla\times\mathbf{A}(\mathbf{r}) \\ &= \frac{i}{c}\sum_{\mathbf{k}\lambda}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}\lambda})\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,[a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}} - a^\dagger_{\mathbf{k}\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}]. \end{aligned} \tag{4.120}$$

It is often convenient to write $\mathbf{E}(\mathbf{r})$ as the sum of two terms:

$$\mathbf{E}^{(+)}(\mathbf{r}) = i\sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{4.121a}$$

$$\mathbf{E}^{(-)}(\mathbf{r}) = [\mathbf{E}^{(+)}(\mathbf{r})]^\dagger. \tag{4.121b}$$

Similarly, for $\mathbf{B}(\mathbf{r})$ we define

$$\mathbf{B}^{(+)}(\mathbf{r}) = \frac{i}{c}\sum_{\mathbf{k}\lambda}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}\lambda})\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,a_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{4.122a}$$

$$\mathbf{B}^{(-)}(\mathbf{r}) = [\mathbf{B}^{(+)}(\mathbf{r})]^\dagger. \tag{4.122b}$$

(In these definitions, the superscript $(+)$ is associated with the annihilation operator a and the superscript $(-)$ is associated with $a^\dagger$. This convention is not universal: some authors relate $(+)$ with $a^\dagger$ and $(-)$ with a.)

In Section 4.1 we called attention to the rotational properties of the electromagnetic field. These features are carried along in the process of quantization. In the context of the photon picture, we say that the photon has an intrinsic spin $S = 1$ with $M_S = \pm 1$; the value $M_S = 0$ is missing on account of the transversality condition. For a given mode, the z axis may be chosen to lie in the direction of propagation, that is, in the direction of the vector $\mathbf{k}$. If the z axis also serves as the axis of quantization, the two possible values of M_S indicate that the spin vector has only two possible orientations—parallel or antiparallel to the photon momentum $\hbar\mathbf{k}$. In the parallel case, the photon is said to have positive *helicity* $(+1)$ or left circular polarization represented by the complex unit polarization vector

$$\hat{\boldsymbol{\varepsilon}}_{\mathbf{k},+1} = -\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}1} + i\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}2}). \tag{4.123a}$$

For the antiparallel case, the photon has negative helicity (-1) or right circular polarization given by the vector

$$\hat{\boldsymbol{\varepsilon}}_{\mathbf{k},-1} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}1} - i\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}2}). \tag{4.123b}$$

The two polarization vectors given above are merely generalizations of the corresponding two vectors in Eq. (4.35).

The photon picture that emerges from the quantization of the electromagnetic field contains a number of singularities that make the theory somewhat less than totally consistent internally. The expression for the energy in Eq. (4.108), for example, contains a zero point energy of $1/2\hbar\omega$ for each mode, even when the photon number n is zero. Since there are an infinite number of modes, the total energy is infinite for every state $|n_1, n_2, \ldots, n_j, \ldots\rangle$, including the vacuum state in Eq. (4.112), which has no photons at all.

The expectation value of the electric field and its square also give rise to unexpected results when viewed from a classical standpoint. Thus, for a single mode with photon number n and frequency ω, the expectation value of $\mathbf{E}(\mathbf{r})$ is

$$\langle n|\mathbf{E}(\mathbf{r})|n\rangle = \langle n|\mathbf{E}^{(+)}(\mathbf{r})|n\rangle + \langle n|\mathbf{E}^{(-)}(\mathbf{r})|n\rangle. \tag{4.124}$$

The two terms on the right are proportional to $\langle n|a|n\rangle$ and $\langle n|a^\dagger|n\rangle$, respectively, both of which are zero as one easily may verify on the basis of Eq. (4.101) or (4.81). Hence,

$$\langle n|\mathbf{E}(\mathbf{r})|n\rangle = 0 \tag{4.125}$$

for all values of the photon number, no matter how large. This result holds for all modes, which means, then, that the expectation value of the electric field in any many-photon state is zero. On the other hand, the expectation value of $E^2(\mathbf{r})$ is proportional to $\langle n|a^\dagger a + aa^\dagger|n\rangle$, which leads to the conclusion that for a single mode

$$\langle n|E^2(\mathbf{r})|n\rangle = \frac{\hbar\omega}{\varepsilon_0 V}\left(n + \frac{1}{2}\right). \tag{4.126}$$

For this case, as for the energy, the expectation value is nonvanishing even when $n = 0$, with the result that for the total field, consisting of an infinite number of modes, the expectation value of $E^2(\mathbf{r})$ is infinite for all many-photon states including the vacuum state.

The same comments apply to the mean square fluctuation (variance), which is defined by

$$\begin{aligned}(\Delta E)^2 &= \langle n|E^2(\mathbf{r})|n\rangle - (\langle n|E(\mathbf{r})|n\rangle)^2 \\ &= \langle n|E^2(\mathbf{r})|n\rangle = \frac{\hbar\omega}{\varepsilon_0 V}\left(n + \frac{1}{2}\right).\end{aligned} \tag{4.127}$$

These and other shortcomings notwithstanding, the quantum theory of radiation is extremely successful; in the low energy regime particularly, which will be our main concern, the singularities are of little consequence and can be ignored in most cases.

4.5 Coherent States

Thus far, the quantum mechanical description of the radiation field has been based on the photon-number formalism. For some theoretical discussions, an alternative basis is preferable. The radiation from a well-stabilized laser oscillator operating in a single mode, for example, is not in a pure photon-number state. To a good approximation, such radiation is in a *coherent* state [5–8] whose properties we now will develop. We retain the same general picture of a radiation field confined within a cubical cavity and described as a superposition of quantized normal modes. Since the latter are independent, the discussion may be simplified by focusing attention on a single mode.

A coherent state $|\alpha\rangle$ is defined as a normalized eigenstate of the annihilation operator,

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{4.128}$$

with

$$\langle\alpha|\alpha\rangle = 1. \tag{4.129}$$

Here $|\alpha\rangle$ is the eigenstate and α is the eigenvalue that may be complex since the annihilation operator is non-Hermitian. Note that we have not imposed an orthogonality condition on two coherent states, $|\alpha\rangle$ and $|\beta\rangle$. The initial objective is to express $|\alpha\rangle$ in terms of the photon-number states $|n\rangle$, that is, to evaluate $\langle n|\alpha\rangle$ in the expansion

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle. \tag{4.130}$$

Since $a|n\rangle = n^{1/2}|n-1\rangle$ and $a|0\rangle = 0$,

$$\begin{aligned}a|\alpha\rangle &= \sum_{n=0}^{\infty} \langle n|\alpha\rangle a|n\rangle = \sum_{n=1}^{\infty} \langle n|\alpha\rangle n^{1/2}|n-1\rangle \\ &= \sum_{n=0}^{\infty} \langle n+1|\alpha\rangle(n+1)^{1/2}|n\rangle.\end{aligned} \tag{4.131}$$

Also, from Eq. (4.130),

$$a|\alpha\rangle = \alpha|\alpha\rangle = \sum_{n=0}^{\infty} \langle n|\alpha\rangle\alpha|n\rangle. \tag{4.132}$$

Equating the two expressions for $a|\alpha\rangle$ yields

$$\sum_n \langle n+1|\alpha\rangle(n+1)^{1/2}|n\rangle = \sum_n \langle n|\alpha\rangle a|n\rangle, \tag{4.133}$$

and upon premultiplying both sides by the photon-number state $\langle m|$ and invoking the orthogonality property $\langle m|n\rangle = \delta_{mn}$, we obtain the recursion

relation

$$\langle n+1|\alpha\rangle = \frac{\langle n|\alpha\rangle}{\sqrt{n+1}}\alpha. \tag{4.134}$$

All the coefficients $\langle n|\alpha\rangle$ may be expressed now in terms of the single coefficient $c_0 \equiv \langle 0|\alpha\rangle$:

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}c_0. \tag{4.135}$$

Substituting in Eq. (4.130),

$$|\alpha\rangle = c_0 \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{4.136}$$

To evaluate c_0 we use the normalization condition $\langle\alpha|\alpha\rangle = 1$ and the orthogonality property of the photon-number states. Thus,

$$\begin{aligned}\langle\alpha|\alpha\rangle &= |c_0|^2 \sum_{m,n} \frac{(\alpha^*)^m\alpha^n}{\sqrt{m!n!}}\langle m|n\rangle = |c_0|^2 \sum_n \frac{(|\alpha|^2)^n}{n!} \\ &= |c_0|^2 e^{|\alpha|^2} = 1\end{aligned} \tag{4.137}$$

or

$$c_0 = e^{-1/2|\alpha|^2}. \tag{4.138}$$

Therefore, the expansion of the coherent state $|\alpha\rangle$ in terms of photon-number states is given by

$$|\alpha\rangle = e^{-1/2|\alpha|^2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-1/2|\alpha|^2+\alpha a^\dagger}|0\rangle, \tag{4.139}$$

where, in the last expression of this equation, we used the relation

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{4.140}$$

The relation $a|\alpha\rangle = \alpha|\alpha\rangle$ implies the conjugate relations

$$\langle\alpha|a^\dagger = \alpha^*\langle\alpha|, \tag{4.141}$$

in which

$$\langle\alpha| = e^{-1/2|\alpha|^2}\sum_n \frac{(\alpha^*)^n}{\sqrt{n!}}\langle n| = \langle 0|e^{-1/2|\alpha|^2+\alpha^* a}. \tag{4.142}$$

From the basic definition of a coherent state as an eigenfunction of the annihilation operator, one obtains the relations

$$\langle\alpha|N|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2 \equiv \langle n\rangle_c, \tag{4.143a}$$

$$\langle\alpha|aa^\dagger|\alpha\rangle = \langle\alpha|N+1|\alpha\rangle = |\alpha|^2 + 1, \tag{4.143b}$$

$$\langle\alpha|N^2|\alpha\rangle = |\alpha|^4 + |\alpha|^2 \equiv \langle n^2\rangle_c \tag{4.143c}$$

$$\langle\alpha|(a^\dagger)^i a^j|\alpha\rangle = (\alpha^*)^i\alpha^j, \tag{4.143d}$$

in which $\langle\alpha|N|\alpha\rangle \equiv \langle n\rangle_c$ is the average number of photons in the coherent state $|\alpha\rangle$.

The expressions for $|\alpha\rangle$ and $\langle\alpha|$ given by Eqs. (4.139) and (4.142) enable us to compute the matrix elements

$$\langle n|\alpha\rangle = e^{-1/2|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}, \qquad \langle\alpha|n\rangle = e^{-1/2|\alpha|^2}\frac{(\alpha^*)^n}{\sqrt{n!}}, \tag{4.144}$$

$$|\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!} = \frac{e^{-\langle n\rangle_c}\langle n\rangle_c^n}{n!} \tag{4.145}$$

The quantity $|\langle n|\alpha\rangle|^2$ is interpretable as the probability of finding n photons in a coherent state $|\alpha\rangle$; therefore, according to the last expression, the probability has a Poissonian distribution.

The coherent states are normalized, as we have seen, but they are not orthogonal; in fact,

$$\begin{aligned}\langle\alpha|\beta\rangle &= e^{-1/2|\alpha|^2}e^{-1/2|\beta|^2}\sum_{m,n}\frac{(\alpha^*)^m\beta^n}{\sqrt{m!n!}}\langle m|n\rangle \\ &= e^{-1/2(|\alpha|^2+|\beta|^2)}\sum_n\frac{(\alpha^*\beta)^n}{\sqrt{n!}} = e^{-1/2(|\alpha|^2+|\beta|^2)}e^{\alpha^*\beta}\end{aligned} \tag{4.146}$$

or

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}. \tag{4.147}$$

Coherent states, although they lack the property of orthogonality, nevertheless satisfy a closure relation of a special type. We write

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\,d^2\alpha = \frac{1}{\pi}\sum_{m,n}\frac{|m\rangle\langle n|}{\sqrt{m!n!}}\int e^{-|\alpha|^2}\alpha^m(\alpha^*)^n\,d^2\alpha, \tag{4.148}$$

where $d^2\alpha$ is an element of area in the complex plane, i.e.,

$$d^2\alpha = d(\mathrm{Re}\,\alpha)\,d(\mathrm{Im}\,\alpha), \tag{4.149a}$$

or, in polar coordinates,

$$\alpha = re^{i\theta}, \qquad d^2\alpha = r\,dr\,de. \tag{4.149b}$$

Then

$$\int e^{-|\alpha|^2}\alpha^m(\alpha^*)^n\,d^2\alpha = \int_0^\infty dr e^{-r^2}r^{m+n+1}\int_0^{2\pi} e^{i(m-n)\theta}\,d\theta$$

$$= 2\pi\int_0^\infty dr e^{-r^2}r^{m+n+1}\,\delta_{mn} = 2\pi\int_0^\infty dr e^{-r^2}r^{2n+1}. \tag{4.150}$$

With a change of variable to $\xi = r^2$, the right side transforms to

$$\pi\int_0^\infty d\xi e^{-\xi}\xi^n = \pi n!. \tag{4.151}$$

Using this relation to evaluate Eq. (4.148) and remembering that $m = n$ as required by the θ integral, we obtain the closure relation for coherent states:

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\,d^2\alpha = \sum_n|n\rangle\langle n| = 1. \tag{4.152}$$

An arbitrary state $|f\rangle$ then may be written

$$|f\rangle = \frac{1}{\pi}\int|\alpha\rangle\langle\alpha|f\rangle\,d^2\alpha. \tag{4.153}$$

As an example of Eq. (4.153) we may obtain an expression for a photon-number state $|n\rangle$ in a coherent state basis, that is, the inverse of Eq. (4.139). Thus,

$$|n\rangle = \frac{1}{\pi}\int|\alpha\rangle\langle\alpha|n\rangle\,d^2\alpha$$

$$= \frac{1}{\pi}\int d^2\alpha e^{-1/2|\alpha|^2}\frac{(\alpha^*)^n}{\sqrt{n!}}|\alpha\rangle, \tag{4.154}$$

where $\langle\alpha|n\rangle$ was obtained from Eq. (4.144). As another example, let $|f\rangle$ be a coherent state $|\beta\rangle$; then, owing to the nonorthogonality of the coherent states, one obtains the rather unusual result of a coherent state expressed as an integral over other coherent states.

$$|\beta\rangle = \frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\beta\rangle\,d^2\alpha = \frac{1}{\pi}\int|\alpha\rangle e^{-1/2(|\alpha|^2+|\beta|^2)+\alpha^*\beta}\,d^2\alpha, \tag{4.155}$$

in which Eq. (4.146) for $\langle\alpha|\beta\rangle$ has been inserted in the integral. With the normalization condition and Eq. (4.147), we also have

$$\langle\beta|\beta\rangle = \frac{1}{\pi}\int|\langle\alpha|\beta\rangle|\,d^2\alpha$$

$$= \frac{1}{\pi}\int e^{-|\alpha-\beta|^2}\,d^2\alpha = 1. \tag{4.156}$$

An arbitrary operator O also may be expressed in a coherent state basis. Two applications of the closure relation in Eq. (4.152) yields

$$O = \frac{1}{\pi}\int O|\alpha\rangle\langle\alpha|\,d^2\alpha$$

$$= \frac{1}{\pi^2}\int d^2\beta\int d^2\alpha|\beta\rangle\langle\beta|O|\alpha\rangle\langle\alpha|. \tag{4.157}$$

But

$$\langle\beta|O|\alpha\rangle = \sum_{m,n}\langle\beta|m\rangle\langle m|O|n\rangle\langle n|\alpha\rangle$$

$$= e^{-1/2(|\alpha|^2+|\beta|^2)}\sum_{m,n}\langle m|O|n\rangle\frac{(\beta^*)^m\alpha^n}{\sqrt{m!n!}}, \tag{4.158}$$

in which, again, we have referred to Eq. (4.144) for $\langle\beta|m\rangle$ and $\langle n|\alpha\rangle$. It is convenient to define

$$R(\alpha,\beta^*) = \langle\beta|O|\alpha\rangle e^{1/2(|\alpha|^2+|\beta|^2)} = \sum_{m,n}\langle m|O|n\rangle\frac{(\beta^*)^m\alpha^n}{\sqrt{m!n!}} \tag{4.159}$$

so that the arbitrary operator O may be written in the form

$$O = \frac{1}{\pi^2}\int d^2\beta\int d^2\alpha R(\alpha,\beta^*)e^{-1/2(|\alpha|^2+|\beta|^2)}|\beta\rangle\langle\alpha|. \tag{4.160}$$

It also is observed that Eq. (4.152) enables us to write

$$\mathrm{Tr}\,O = \sum_n\langle n|O|n\rangle = \frac{1}{\pi}\int d^2\alpha\sum_n\langle n|O|\alpha\rangle\langle\alpha|n\rangle$$

$$= \frac{1}{\pi}\int d^2\alpha\sum_n\langle\alpha|n\rangle\langle n|O|\alpha\rangle = \frac{1}{\pi}\int d^2\alpha\langle\alpha|O|\alpha\rangle. \tag{4.161}$$

A coherent state will evolve in time according to the general relation

$$|\alpha(t)\rangle = e^{-i\mathscr{H}t/\hbar}|\alpha\rangle \tag{4.162}$$

where $|\alpha\rangle \equiv |\alpha(O)\rangle$ and $\mathscr{H} = \hbar\omega(N + 1/2)$. Using the expansion (4.139) for the coherent state in terms of the photon-number states,

$$|\alpha(t)\rangle = \sum_n e^{-1/2|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega(N+1/2)t}|n\rangle$$

$$= e^{-i\omega t/2} \sum_n \frac{(\alpha e^{-i\omega t})}{\sqrt{n!}} e^{-1/2|\alpha|^2}|n\rangle = e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle, \tag{4.163}$$

from which it may be concluded that a coherent state remains coherent as time progresses.

According to Eq. (4.125), we found that the expectation value of the electric field in a photon-number state vanishes. In a coherent state, however, the expectation value of the field in a single mode is

$$\langle\alpha|\mathbf{E}(\mathbf{r})|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,\hat{\varepsilon}(\langle\alpha|a|\alpha\rangle e^{i\mathbf{k}\cdot\mathbf{r}} - \langle\alpha|a^\dagger|\alpha\rangle e^{-i\mathbf{k}\cdot\mathbf{r}})$$

$$= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,\hat{\varepsilon}(\alpha e^{i\mathbf{k}\cdot\mathbf{r}} - \alpha^* e^{-i\mathbf{k}\cdot\mathbf{r}}), \tag{4.164}$$

which is not automatically zero. Indeed, $\langle\alpha|\mathbf{E}(\mathbf{r})|\alpha\rangle$ represents a classical traveling plane wave whose phase is that of α and whose amplitude is proportional to the absolute value of α which, according to Eq. (4.143a), is the square root of the average number of photons. Neither does the expectation

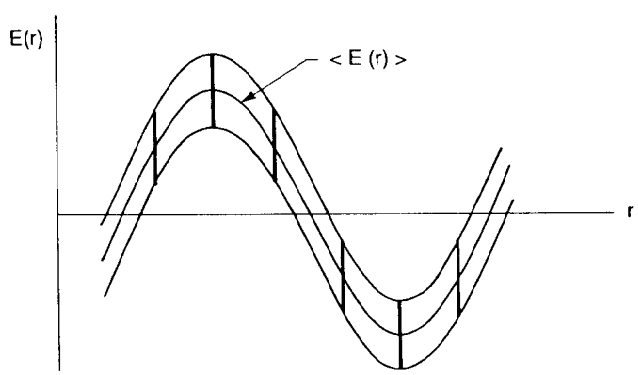

FIGURE 4.1 The middle curve is the expectation value $\langle E(r)\rangle = \langle\alpha|E(r)|\alpha\rangle$ of a (scalar) electric field in a coherent state $|\alpha\rangle$. The vertical bars indicate the root mean square fluctuation ΔE where $(\Delta E)^2 = \langle\alpha|E^2(r)|\alpha\rangle - (\langle\alpha|E(r)|\alpha\rangle)^2$. The rms fluctuation ΔE is independent of r and α.

value of $E^2(\mathbf{r})$ vanish; thus, when $\mathbf{k}\cdot\mathbf{r} \ll 1$, for example,

$$\langle\alpha|E^2(\mathbf{r})|\alpha\rangle = \frac{\hbar\omega}{2\varepsilon_0 V}(1 - (\alpha - \alpha^*)^2), \tag{4.165}$$

and the mean square fluctuation for this case is

$$(\Delta E)^2 = \langle\alpha|E^2(\mathbf{r})|\alpha\rangle - (\langle\alpha|\mathbf{E}(\mathbf{r})|\alpha\rangle)^2 = \frac{\hbar\omega}{2\varepsilon_0 V}. \tag{4.166}$$

This result (Fig. 4.1), which is finite and independent of α, contrasts sharply with the mean square fluctuation in the photon-number basis (Eq. (4.127)), which tends to infinity with increasing photon number n. Note, also, that $(\Delta E)^2$ for a coherent state (Eq. (4.166)) is equal to $(\Delta E)^2$ for a photon-number state (Eq. (4.127)) only when $n = 0$. That is, the quantum fluctuations in a coherent state are precisely the zero-point fluctuations of the vacuum.

When the electric field is written in the Heisenberg representation in order to include the time-dependence, it is seen that there is a close correspondence between a single mode coherent state and a classical plane wave. The multi-mode coherent state

$$|\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots\rangle \equiv |\{\alpha_i\}\rangle = \prod_i |\alpha_i\rangle \tag{4.167}$$

corresponds to a superposition of plane waves.

4.6 Displacement Operator and Characteristic Functions

This section is devoted to the derivation of a number of results which reappear in connection with squeezed states (Section 4.9) and with photon statistics (Section 5.13).

Coherent states may be discussed in terms of a *translation* or *displacement* operator $D(\alpha)$ [8] defined by

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}. \tag{4.168}$$

Although $D(\alpha)$ cannot be written as a simple product of $e^{\alpha a^\dagger}$ and $e^{-\alpha^* a}$ a useful theorem exists known as the *disentangling theorem*, which is a special case of the Campbell-Baker-Hausdorff formula [3]. The theorem states that it A and B are two noncommuting operators that satisfy the conditions

$$[A, [A, B]] = [B, [A, B]] = 0, \tag{4.169}$$

then

$$e^{A+B} = e^A e^B e^{-1/2[A,B]} = e^B e^A e^{1/2[A,B]}. \tag{4.170}$$

Letting $A = \alpha a^\dagger$, $B = -\alpha^* a$, and noting that $[a, a^\dagger] = 1$, we have

$$\begin{aligned} [A, B] &= [\alpha a^\dagger, -\alpha^* a] = |\alpha|^2, \\ [A, [A, B]] &= [\alpha a^\dagger, |\alpha|^2] = 0, \\ [B, [A, B]] &= [-\alpha^* a, |\alpha|^2] = 0. \end{aligned} \tag{4.171}$$

It now becomes possible to express $D(\alpha)$ in the form

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-1/2|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a} \tag{4.172a}$$

$$= e^{1/2|\alpha|^2} e^{-\alpha^* a} e^{\alpha a^\dagger}. \tag{4.172b}$$

Note that in Eq. (4.172a), the product of the exponentials is in normal order since the creation operators will appear to the left of annihilation operators when the exponentials are written as power series; similarly, Eq. (4.172b) is in antinormal order. The Hermitian conjugate of Eq. (4.172a) is

$$D^\dagger(\alpha) = e^{-1/2|\alpha|^2} e^{-\alpha a^\dagger} e^{\alpha^* a}, \tag{4.173}$$

and when this expression is premultiplied by $D(\alpha)$ from Eq. (4.172b), we have

$$D(\alpha) D^\dagger(\alpha) = 1. \tag{4.174}$$

Therefore, $D(\alpha)$ is a unitary operator:

$$\begin{aligned} D^\dagger(\alpha) = D^{-1}(\alpha) &= e^{-1/2|\alpha|^2} e^{-\alpha a^\dagger} e^{\alpha^* a} = e^{1/2|\alpha|^2} e^{\alpha^* a} e^{-\alpha a^\dagger} \\ &= e^{-\alpha a^\dagger + \alpha^* a} = D(-\alpha). \end{aligned} \tag{4.175}$$

Another general operator theorem, derivable from Eq. (4.170), states that if A and B are two noncommuting operators,

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots. \tag{4.176}$$

In the present context, let

$$A = -\alpha a^\dagger + \alpha^* a, \qquad B = a. \tag{4.177}$$

Then,

$$\begin{aligned} e^A &= e^{-\alpha a^\dagger + \alpha^* a} = D^{-1}(\alpha), \\ e^{-A} &= e^{\alpha a^\dagger - \alpha^* a} = D(\alpha), \\ [A, B] &= -\alpha[a^\dagger, a] = \alpha, \end{aligned} \tag{4.178}$$

and all higher order commutators vanish. Thus,

$$D^{-1}(\alpha) a D(\alpha) = a + \alpha, \qquad D^{-1}(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^*, \tag{4.179a}$$

or

$$D(\alpha) a D^{-1}(\alpha) = a - \alpha, \qquad D(\alpha) a^\dagger D^{-1}(\alpha) = a^\dagger - \alpha^*. \tag{4.179b}$$

The reason for calling $D(\alpha)$ a translation or displacement operator now becomes apparent. When the annihilation and creation operators a and $a^\dagger$ are subjected to a unitary transformation by $D(\alpha)$, as in Eq. (4.179), a and $a^\dagger$ are augmented by α and α^* respectively.

The coherent state has been defined by the basic relation $a|\alpha\rangle = \alpha|\alpha\rangle$. Then, in view of Eq. (4.179), we may write

$$\begin{aligned} D^{-1}(\alpha) a D(\alpha) D^{-1}(\alpha)|\alpha\rangle &= D^{-1}(\alpha) a|\alpha\rangle \\ &= \alpha D^{-1}(\alpha)|\alpha\rangle = (a + \alpha) D^{-1}(\alpha)|\alpha\rangle \end{aligned} \tag{4.180}$$

which means that

$$a D^{-1}(\alpha)|\alpha\rangle = 0. \tag{4.181}$$

Since $a|0\rangle = 0$, it is possible to make the identification

$$D^{-1}(\alpha)|\alpha\rangle = |0\rangle \qquad \text{or} \qquad |\alpha\rangle = D(\alpha)|0\rangle. \tag{4.182}$$

In other words, a coherent state $|\alpha\rangle$ can be generated from the vacuum state $|0\rangle$ by means of the operator $D(\alpha)$ or, alternatively, a coherent state $|\alpha\rangle$ may be transformed to the vacuum state by $D^{-1}(\alpha)$. Indeed, the various properties of coherent states, derived in the previous section starting with Eq. (4.128), may be derived just as easily from Eq. (4.182) which then serves as the definition of a coherent state.

An alternative verification of Eq. (4.181) may be obtained by referring to the power series expansion of an exponential operator. Thus,

$$e^{-\alpha^* a}|0\rangle = \sum_n \frac{(-\alpha^* a)^n}{n!}|0\rangle = |0\rangle, \tag{4.183}$$

since only the term with $n = 0$ is nonvanishing, and

$$e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle = e^{\alpha a^\dagger}|0\rangle = \sum_n \frac{(\alpha a^\dagger)^n}{n!}|0\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{4.184}$$

Therefore, from Eqs. (4.172), (4.184), and (4.180),

$$\begin{aligned} D(\alpha)|0\rangle &= e^{-1/2|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle = e^{-1/2|\alpha|^2} \sum_n \frac{(\alpha a^\dagger)^n}{n!}|0\rangle \\ &= e^{-1/2|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle. \end{aligned} \tag{4.185}$$

as in Eq. (4.182).

Characteristic functions are defined [8] as follows:

$$\chi(\lambda) = \mathrm{Tr}\{\rho D(\lambda)\} = \mathrm{Tr}\{\rho e^{\lambda a^\dagger - \lambda^* a}\}, \tag{4.186a}$$

$$\chi_n(\lambda) = \mathrm{Tr}\{\rho e^{\lambda a^\dagger} e^{-\lambda^* a}\}, \tag{4.186b}$$

$$\chi_a(\lambda) = \mathrm{Tr}\{\rho e^{-\lambda^* a} e^{\lambda a^\dagger}\}, \tag{4.186c}$$

where $\chi_n(\lambda)$ is in normal order and $\chi_a(\lambda)$ is in antinormal order; λ is a complex variable. The three functions are related by the disentangling theorem (4.170). Thus, based on Eq. (4.172), we have

$$\chi(\lambda) = \chi_a(\lambda) e^{1/2|\lambda|^2} = \chi_n(\lambda) e^{-1/2|\lambda|^2}. \tag{4.187}$$

The characteristic functions have many interesting properties. It is found, for example, that

$$\frac{\partial}{\partial\lambda}\frac{\partial}{\partial(-\lambda^*)}\chi_n(\lambda)\bigg|_{\lambda=0} = \mathrm{Tr}\{\rho a^\dagger e^{\lambda a^\dagger} e^{-\lambda^* a} a\}\bigg|_{\lambda=0}$$
$$= \mathrm{Tr}\{\rho a^\dagger a\} = \langle a^\dagger a\rangle. \tag{4.188}$$

This is generalized easily to

$$\frac{\partial^n}{\partial\lambda^n}\frac{\partial^m}{\partial(-\lambda^*)^m}\chi_n(\lambda)\bigg|_{\lambda=0} = \mathrm{Tr}\{\rho(a^\dagger)^n a^m\}. \tag{4.189a}$$

In similar fashion,

$$\frac{\partial^n}{\partial\lambda^n}\frac{\partial^m}{\partial(-\lambda^*)^m}\chi_a(\lambda)\bigg|_{\lambda=0} = \mathrm{Tr}\{\rho a^m (a^\dagger)^n. \tag{4.189b}$$

By virtue of the theorem (4.161), the characteristic functions are expressible in the form of integrals. In particular,

$$\chi_a(\lambda) = \mathrm{Tr}\{\rho e^{-\lambda^* a} e^{\lambda a^\dagger}\} = \mathrm{Tr}\{e^{\lambda a^\dagger}\rho e^{-\lambda^* a}\}$$
$$= \frac{1}{\pi}\int d^2\alpha \langle\alpha|e^{\lambda a^\dagger}\rho e^{-\lambda^* a}|a\rangle = \frac{1}{\pi}\int d^2\alpha\langle\alpha|\rho|\alpha\rangle e^{\lambda\alpha^* - \lambda^*\alpha}. \tag{4.190}$$

The last integral is but a thinly disguised expression for the two-dimensional Fourier transform of $\langle\alpha|\rho|\alpha\rangle$. To appreciate this fact, let the complex quantities α and λ be written

$$\alpha = \frac{(q + ip)}{\sqrt{2}}, \qquad \lambda = \frac{(x + ik)}{\sqrt{2}}. \tag{4.191}$$

Then,

$$\langle\alpha|\rho|\alpha\rangle = \langle q, p|\rho|q, p\rangle, \qquad \lambda\alpha^* - \lambda^*\alpha = i(kq - xp),$$
$$d^2\alpha = d(\mathrm{Re}\,\alpha)\,d(\mathrm{Im}\,\alpha) = \frac{1}{2}dq\,dp, \tag{4.192}$$

and

$$\chi_a(x, k) = \frac{1}{2\pi}\int \langle q, p|\rho|q, p\rangle e^{i(kq - xp)}\,dq\,dp. \tag{4.193}$$

The inverse Fourier transform is

$$\langle q, p|\rho|q, p\rangle = \frac{1}{2\pi}\int \chi_a(x, k) e^{i(xp - kq)}\,dx\,dk, \tag{4.194}$$

which reverts to

$$\langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi}\int \chi_a(\lambda) e^{\lambda^*\alpha - \lambda\alpha^*}\,d\lambda. \tag{4.195}$$

We now shall develop a number of theorems pertaining to coherent states. Referring to Eq. (4.139) and (4.142),

$$|\alpha\rangle\langle\alpha| = e^{(\alpha a^\dagger - 1/2|\alpha|^2)}|0\rangle\langle 0|e^{(\alpha^* a - 1/2|\alpha|^2)}$$
$$= e^{-\alpha^*\alpha} e^{\alpha a^\dagger}|0\rangle\langle 0|e^{\alpha^* a}. \tag{4.196}$$

Hence,

$$\frac{\partial}{\partial\alpha}|\alpha\rangle\langle\alpha| = (a^\dagger - \alpha^*)|\alpha\rangle\langle\alpha| \tag{4.197a}$$

or

$$a^\dagger|\alpha\rangle\langle\alpha| = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|. \tag{4.197b}$$

The Hermitian adjoint yields

$$|\alpha\rangle\langle\alpha|a = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|. \tag{4.198}$$

Note that if $\alpha = x + iy$ and $\alpha^* = x - iy$,

$$\frac{\partial}{\partial\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial\alpha^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{4.199}$$

since

$$a|\alpha\rangle\langle\alpha| = \alpha|\alpha\rangle\langle\alpha|, \qquad |\alpha\rangle\langle\alpha|a^\dagger = \alpha^*|\alpha\rangle\langle\alpha|, \tag{4.200}$$

we have

$$a^\dagger a|\alpha\rangle\langle\alpha| = \left(\alpha\frac{\partial}{\partial\alpha} + |\alpha|^2\right)|\alpha\rangle\langle\alpha|. \tag{4.201}$$

Similarly,

$$|\alpha\rangle\langle\alpha|a^\dagger a = \left(a^*\frac{\partial}{\partial\alpha^*} + |\alpha|^2\right)|\alpha\rangle\langle\alpha|. \tag{4.202}$$

These relations may be extended to arbitrary products of a and $a^\dagger$; for example,

$$|\alpha\rangle\langle\alpha|a^\dagger aa = \alpha\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|a = \alpha\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)^2|\alpha\rangle\langle\alpha|, \tag{4.203a}$$

$$\begin{aligned}|\alpha\rangle\langle\alpha|aa^\dagger a &= \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|a^\dagger a \\ &= \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\alpha^*\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|.\end{aligned} \tag{4.203b}$$

Thus, if $F(a, a^\dagger)$ is a function of a and $a^\dagger$ expressible as a power series in the operators, and $|\psi\rangle$ is an arbitrary wave function, we have the *classical-quantum* correspondence

$$\langle\alpha|F(a, a^\dagger)|\psi\rangle = F\left(\frac{\partial}{\partial\alpha^*} + \alpha, \alpha^*\right)\langle\alpha|\psi\rangle \tag{4.204}$$

whereby the annihilation operator a is replaced by $(\partial/\partial\alpha^*) + \alpha$, and the creation operator $a^\dagger$ is replaced by α^*. What Eq. (4.204) reveals is that matrix elements of the quantum mechanical operators a and $a^\dagger$ may be computed by c number operations, i.e., operations that do not involve quantum mechanical operators.

From the relation

$$a|\alpha\rangle\langle\alpha|a^\dagger = |\alpha|^2|\alpha\rangle\langle\alpha| \tag{4.205}$$

one obtains

$$a^\dagger a|\alpha\rangle\langle\alpha| - 2a|\alpha\rangle\langle\alpha|a^\dagger + |\alpha\rangle\langle\alpha|a^\dagger a = \left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|. \tag{4.206}$$

Several additional relations that may be verified by similar methods are the following:

$$aa^\dagger|\alpha\rangle\langle\alpha| = (a^\dagger a + 1)|\alpha\rangle\langle\alpha| = \left(\alpha\frac{\partial}{\partial\alpha} + |\alpha|^2 + 1\right)|\alpha\rangle\langle\alpha|, \tag{4.207a}$$

$$|\alpha\rangle\langle\alpha|aa^\dagger = |\alpha\rangle\langle\alpha|(a^\dagger a + 1) = \left(\alpha^\dagger\frac{\partial}{\partial\alpha^*} + |\alpha|^2 + 1\right)|\alpha\rangle\langle\alpha|, \tag{4.207b}$$

$$\begin{aligned}a^\dagger|\alpha\rangle\langle\alpha|a &= \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|a = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha| \\ &= \left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \alpha^*\frac{\partial}{\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + |\alpha|^2 + 1\right)|\alpha\rangle\langle\alpha|.\end{aligned} \tag{4.207c}$$

From these we obtain

$$\begin{aligned}&aa^\dagger|\alpha\rangle\langle\alpha| - 2a^\dagger|\alpha\rangle\langle\alpha|a + |\alpha\rangle\langle\alpha|aa^\dagger \\ &\quad = -\left(2\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|.\end{aligned} \tag{4.208}$$

4.7 Statistical Properties of Photon-Number States

The statistical properties of the radiation field are most conveniently treated by means of the density matrix formalism. In the photon number basis that contains the closure property

$$\sum_n |n\rangle\langle n| = 1, \tag{4.209}$$

the density operator may be written

$$\rho = \sum_{mn} |m\rangle\langle m|\rho|n\rangle\langle n|. \tag{4.210}$$

When the radiation field is under thermal equilibrium with the walls of the cavity, ρ becomes the thermal density operator ρ_0 whose general form is given by Eq. (2.226). For a single mode, with $\mathscr{H} = N + 1/2 \equiv a^\dagger a + 1/2$,

$$\frac{1}{Z}\langle m|e^{-\beta\mathscr{H}}|n\rangle = \frac{1}{Z}\langle m|e^{-\beta\hbar\omega(N+1/2)}|n\rangle = \frac{1}{Z}e^{-\beta\hbar\omega(n+1/2)}\delta_{mn}, \tag{4.211}$$

$$\begin{aligned}Z = \mathrm{Tr}\{e^{-\beta\mathscr{H}}\} &= \sum_n \langle n|e^{-\beta\hbar\omega(N+1/2)}|n\rangle = e^{-\beta\hbar\omega/2}\sum_n e^{-\beta n\hbar\omega} \\ &= \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}.\end{aligned} \tag{4.212}$$

We have

$$\langle m|\rho_0|n\rangle = (1 - e^{-\beta\hbar\omega})e^{-\beta n\hbar\omega}\delta_{mn}, \tag{4.213}$$

and

$$\rho_0 = (1 - e^{-\beta\hbar\omega})\sum_n e^{-\beta n\hbar\omega}|n\rangle\langle n|. \tag{4.214}$$

The diagonal matrix element $\langle n|\rho_0|n\rangle$ gives the probability of finding n photons at the frequency ω when the electromagnetic field is in thermal equilibrium at the temperature T.

Having obtained an expression for the thermal density operator of an electromagnetic field in a cavity, we now may calculate various average properties. The first important quantity is the average number of photons:

$$\begin{aligned}\langle n\rangle &= \mathrm{Tr}\{\rho_0 N\} = (1-e^{-\beta\hbar\omega})\mathrm{Tr}\left\{\sum_n e^{-n\beta\hbar\omega}|n\rangle\langle n|N\right\}\\ &= (1-e^{-\beta\hbar\omega})\sum_{m,n} e^{-n\beta\hbar\omega}\langle m|n\rangle\langle n|N|m\rangle\\ &= (1-e^{-\beta\hbar\omega})\sum_n ne^{-n\beta\hbar\omega}. \end{aligned} \tag{4.215}$$

Upon setting $x = \beta\hbar\omega$,

$$\begin{aligned}\sum_n ne^{-n\beta\hbar\omega} &= \sum_n ne^{-nx} = -\frac{\partial}{\partial x}\sum_n e^{-nx} = -\frac{\partial}{\partial x}\frac{1}{1-e^{-x}}\\ &= \frac{e^{-x}}{(1-e^{-x})^2} = \frac{e^{-\beta\hbar\omega}}{(1-e^{\beta\hbar\omega})^2}. \end{aligned} \tag{4.216}$$

Thus,

$$\langle n\rangle = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega}-1} = \begin{cases} 1/\beta\hbar\omega = kT/\hbar\omega, & (kT \gg \hbar\omega) \quad (4.217\text{a})\\ e^{-\beta\hbar\omega} = e^{-\hbar\omega/kT}, & (kT \ll \hbar\omega). \quad (4.217\text{b})\end{cases}$$

At room temperature, the average number of photons in the optical region of the spectrum is extremely small ($\sim 10^{-40}$) but grows rapidly for longer wavelengths ($\langle n\rangle \simeq 1$ for $\lambda = 10 - 100\ \mu$m). For microwaves, $\langle n\rangle \gg 1$; to obtain $\langle n\rangle \simeq 1$ in the visible region, a temperature of $\sim 10^4$ K is required. Note also that $\langle n\rangle \to 0$ when $T \to 0$, independent of frequency.

Correlation functions under thermal equilibrium were described in Section 2.8. For the creation and annihilation operators, we have

$$\langle a_{\mathrm{H}}^\dagger(\tau)a\rangle = e^{i\omega\tau}\langle a^\dagger a\rangle = e^{i\omega\tau}\langle n\rangle, \tag{4.218a}$$

$$\langle a_{\mathrm{H}}(\tau)a^\dagger\rangle = e^{-i\omega\tau}\langle aa^\dagger\rangle = e^{-i\omega\tau}[\langle a^\dagger a\rangle + 1] = e^{-i\omega\tau}[\langle n\rangle + 1], \tag{4.218b}$$

$$\langle a^\dagger a_{\mathrm{H}}(\tau)\rangle = e^{-i\omega\tau}\langle a^\dagger a\rangle = e^{-i\omega\tau}\langle n\rangle, \tag{4.218c}$$

$$\langle aa_{\mathrm{H}}^\dagger(\tau)\rangle = e^{i\omega\tau}\langle aa^\dagger\rangle = e^{i\omega\tau}[\langle n\rangle + 1], \tag{4.218d}$$

$$\langle a_{\mathrm{H}}^\dagger(\tau)a^\dagger\rangle = \langle a_{\mathrm{H}}(\tau)a\rangle = 0, \tag{4.218e}$$

where the conversion of $a_{\mathrm{H}}^\dagger$ and a_{H} in the Heisenberg representation to $a^\dagger$ and a in the Schrödinger representation was made according to Eq. (4.83) and $\langle n\rangle$ is given by Eq. (4.217). We note that a and $a^\dagger$ belong to the same radiation mode; all correlation functions vanish when a and $a^\dagger$ belong to different modes.

The expression for the average number of photons may be used to derive a temperature-independent form for the density operator. Combining Eqs. (4.214) and (4.217),

$$\rho_0 = \frac{1}{1+\langle n\rangle}\sum_n\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n |n\rangle\langle n|, \tag{4.219a}$$

$$\langle n|\rho_0|n\rangle = \frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n. \tag{4.219b}$$

The diagonal matrix element, when written in this form, gives the probability of finding n photons in a mode whose average number of photons is $\langle n\rangle$. Although these expressions were derived specifically for the case of thermal equilibrium, they are, nevertheless, also applicable to the general case of a random (chaotic) photon distribution, which may have been produced by a nonthermal randomizing process.

For thermal photon distributions, the mean square fluctuation (variance) is given by

$$(\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2 \tag{4.220}$$

where

$$\begin{aligned}\langle n^2\rangle &= \mathrm{Tr}\{\rho_0 N^2\}\\ &= (1-e^{-\beta\hbar\omega})\mathrm{Tr}\left\{\sum_n e^{-n\beta\hbar\omega}|n\rangle\langle n|N^2\right\}. \end{aligned} \tag{4.221}$$

But

$$\begin{aligned}\mathrm{Tr}\left\{\sum_n e^{-n\beta\hbar\omega}|n\rangle\langle n|N^2\right\} &= \sum_{m,n} e^{-n\beta\hbar\omega}\langle m|n\rangle\langle n|N^2|m\rangle\\ &= \sum_n e^{-n\beta\hbar\omega}\langle n|N^2|n\rangle = \sum_n n^2 e^{-n\beta\hbar\omega}. \end{aligned} \tag{4.222}$$

Again, setting $x = \beta\hbar\omega$ and writing

$$\sum_n n^2 e^{-nx} = \frac{\partial^2}{\partial x^2}\sum_n e^{-nx} = \frac{\partial^2}{\partial x^2}\frac{1}{1-e^{-x}}, \tag{4.223}$$

one finds, with the expression for $\langle n\rangle$ given by Eq. (4.217),

$$\langle n^2\rangle = \frac{e^{\beta\hbar\omega}+1}{(e^{\beta\hbar\omega}-1)^2} = \langle n\rangle + 2\langle n\rangle^2, \tag{4.224}$$

$$(\Delta n)^2 = e^{\beta\hbar\omega}\langle n\rangle^2 = \langle n\rangle(1+\langle n\rangle). \tag{4.225}$$

$$\eta \equiv \frac{\Delta n}{\langle n\rangle} = \frac{[\langle n\rangle(1+\langle n\rangle)]^{1/2}}{\langle n\rangle} = \begin{cases} 1, & \langle n\rangle \gg 1\\ \langle n\rangle^{-1/2}, & \langle n\rangle \ll 1.\end{cases} \tag{4.226}$$

The quantity η is known as the *relative uncertainty*. Under thermal equilibrium, η approaches infinity as $\langle n\rangle \to 0$. It is the uncertainty (or fluctuations) in this regime that is associated with quantum noise.

The *Planck radiation law* follows directly from the average photon number. All that is required is to multiply the average energy $\hbar\omega\langle n\rangle$ by the number of modes per unit interval in ω in a unit volume, taking into account the two independent directions of polarization. From Eqs. (4.43) and (4.217) we have

$$U(\omega) = 2\hbar\omega\langle n\rangle \frac{\omega^2}{2\pi^2c^3} = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\beta\hbar\omega}-1}. \tag{4.227}$$

$U(\omega)$ is the average energy per unit volume per unit interval in ω, for a distribution of photons at thermal equilibrium; it is also known as the "black body distribution." It is of interest to observe that, despite the presence of the zero-point energy in the Hamiltonian (Eq. (4.97)), $U(\omega)$ is finite at all temperature and vanishes when $T \to 0$. If the unit interval in ω is replaced by unit intervals in $\nu(=\omega/2\pi)$ or $\lambda(=c/\nu)$, the relations among the distributions are

$$U(\omega)\,d\omega = U(\nu)\,d\nu = U(\lambda)\,d\lambda \tag{4.228}$$

or

$$U(\omega) = 2\pi U(\nu) = \frac{(2\pi)^3c}{\omega^3}U(\lambda). \tag{4.229a}$$

One then derives

$$U(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\beta h\nu}-1}, \tag{4.229b}$$

$$U(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\beta hc/\lambda}-1}, \tag{4.229c}$$

where $U(\nu)$ is the average energy per unit volume per unit interval in ν, $U(\lambda)$ is the average energy per unit volume per unit interval in λ, and $h = 2\pi\hbar$. A plot of $U(\lambda)$ for $T = 2000$, 3000, and 4000 K is shown in Fig. 4.2.

In the high-temperature limit, with $\langle n\rangle$ as in Eq. (4.217a), we obtain the *Rayleigh limit*

$$U(\omega) = \frac{\omega^2}{\pi^2c^3\beta}, \qquad \left(\frac{1}{\beta} = kT \gg \hbar\omega\right), \tag{4.230}$$

whereas in the low-temperature limit, with $\langle n\rangle$ given by Eq. (4.217b),

$$U(\omega) = \frac{\hbar\omega^3}{\pi^2c^3}e^{-\beta\hbar\omega}. \tag{4.231}$$

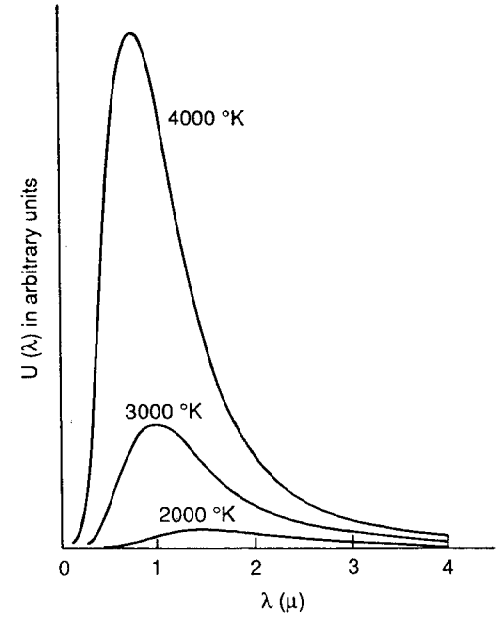

FIGURE 4.2 Curves of black-body radiation at $T = 2000°K$, $3000°K$ and $4000°K$.

The average energy per unit volume is obtained by integrating $U(\omega)$ over all frequencies:

$$U = \int_0^\infty U(\omega)\,d\omega = \frac{\hbar}{\pi^2c^3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\,d\omega$$
$$= \frac{\pi^2}{15\beta^4c^3\hbar^3} = 7.56\times10^{-16}\ \mathrm{K}^4, \qquad (\mathrm{J/m^3}). \tag{4.232}$$

The average number of photons per unit volume N_V is

$$N_V = \int_0^\infty \frac{U(\omega)}{\hbar\omega}\,d\omega = \int_0^\infty \langle n\rangle\frac{\omega^2}{\pi^2c^3}\,d\omega = \frac{1}{\pi^2c^3}\int_0^\infty \frac{\omega^2}{e^{\beta\hbar\omega}}\,d\omega$$
$$= 2.028\times10^7\ \mathrm{K}^3, \qquad (\mathrm{photons/m^3}). \tag{4.233}$$

The ratio U/N_V is the average energy per photon

$$E = \frac{U}{N_V} = 3.73\times10^{-23}\ \mathrm{K}, \qquad (\mathrm{J}). \tag{4.234}$$

We also may compute the average number of photons associated with a beam produced by a thermal source. Since the general relation between energy

density and intensity is $I = cU$, we have, from Eq. (4.227),

$$\langle n\rangle = \frac{\pi^2 c^3}{\hbar\omega^3} U(\omega) = \frac{\pi^2 c^2}{\hbar\omega^3} I(\omega) = 8.41 \times 10^{51} \frac{I(\omega)}{\omega^3}. \tag{4.235}$$

Here, $I(\omega)$ is the intensity (Jm^{-2}) per unit interval in ω.

4.8 Statistical Properties of Coherent States

The expression for a general operator in a coherent state basis (Eq. (4.160)) may be applied to the density operator

$$\rho = \frac{1}{\pi^2}\int d^2\beta \int d^2\alpha |\beta\rangle\langle\beta|\rho|\alpha\rangle\langle\alpha|$$

$$= \frac{1}{\pi^2}\int d^2\beta \int d^2\alpha R(\alpha,\beta^*) e^{-1/2(|\alpha|^2+|\beta|^2)} |\beta\rangle\langle\alpha|, \tag{4.236}$$

where, according to Eq. (4.159),

$$R(\alpha,\beta^*) = \sum_{m,n} \langle m|\rho|n\rangle \frac{(\beta^*)^m \alpha^n}{\sqrt{m!n!}}, \tag{4.237}$$

and

$$\langle\beta|\rho|\alpha\rangle = e^{-1/2(|\alpha|^2+|\beta|^2)} R(\alpha,\beta^*). \tag{4.238}$$

Furthermore, from Eq. (4.161) we have

$$\mathrm{Tr}\,\rho = \frac{1}{\pi}\int d^2\alpha \langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi}\int R(\alpha,\alpha^*) e^{-|\alpha|^2} d^2\alpha = 1. \tag{4.239}$$

Under thermal equilibrium, $R(\alpha,\beta^*)$ becomes

$$R(\alpha,\beta^*) = \sum_{n,n} \langle m|\rho_0|n\rangle \frac{(\beta^*)^m \alpha^n}{\sqrt{m!n!}}. \tag{4.240}$$

Using Eqs. (4.213) and (4.217), the special forms of $R(\alpha,\beta^*)$ under thermal equilibrium are

$$R(\alpha,\beta^*) = (1 - e^{-\beta\hbar\omega}) \sum_n \frac{(\beta^*\alpha)^n}{n!} e^{-n\beta\hbar\omega} = \frac{1}{1+\langle n\rangle} \sum_n \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n \frac{(\beta^*\alpha)^n}{n!} \tag{4.241a}$$

$$= \frac{1}{1+\langle n\rangle} \exp\left(\frac{\langle n\rangle}{1+\langle n\rangle}\beta^*\alpha\right). \tag{4.241b}$$

These expressions may be inserted into Eq. (4.238) to obtain the matrix element of the thermal density operator in a coherent state basis. In particular, the diagonal matrix element is

$$\langle\alpha|\rho_0|\alpha\rangle = (1 - e^{-\beta\hbar\omega}) e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} e^{-n\beta\hbar\omega}$$

$$= \frac{1}{1+\langle n\rangle} \exp\left(-\frac{|\alpha|^2}{1+\langle n\rangle}\right). \tag{4.242}$$

The expression for the density operator given by Eq. (4.236) is the general form of the density operator in a coherent state basis. It may be simplified, under certain conditions, by defining a new quantity $P(\alpha)$ such that

$$\rho = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|. \tag{4.243}$$

This is known as the *P-representation* or the *diagonal representation* of the density operator [7, 8]. Since the condition $\mathrm{Tr}\,\rho = 1$ must be satisfied, we have

$$\mathrm{Tr}\,\rho = \sum_n \langle n|\rho|n\rangle = \int d^2\alpha P(\alpha) \sum_n \langle n|\alpha\rangle\langle\alpha|n\rangle = 1. \tag{4.244}$$

But

$$\sum_n \langle n|\alpha\rangle\langle\alpha|n\rangle = \sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle = \langle\alpha|\alpha\rangle = 1; \tag{4.245}$$

hence, we arrive at the normalization property of the P-representation

$$\int P(\alpha)\, d^2\alpha = 1. \tag{4.246}$$

For a general density operator ρ written in terms of $P(\alpha)$, as in Eq. (4.243), the diagonal element is

$$\langle\beta|\rho|\beta\rangle = \int d^2\alpha P(\alpha) \langle\beta|\alpha\rangle\langle\alpha|\beta\rangle = \int d^2\alpha P(\alpha) e^{-|\alpha-\beta|^2}, \tag{4.247}$$

in which the last expression is based on Eq. (4.147).

These properties suggest that $P(\alpha)$ behaves like an ordinary probability density, which is often the case; but $P(\alpha)$ also may become highly singular or negative. Hence, it cannot be regarded as a legitimate probability density under all circumstances. As an example, a pure coherent state consistency between $\rho = |\alpha\rangle\langle\alpha|$ and Eq. (4.243) requires that

$$P(\alpha) = \delta^{(2)}(\alpha - \alpha') = \delta[\mathrm{Re}(\alpha - \alpha')]\, \delta[\mathrm{Im}(\alpha - \alpha')]. \tag{4.248}$$

In terms of $P(\alpha)$, the ensemble average of an operator O is

$$\langle O\rangle = \mathrm{Tr}\{\rho O\} = \sum_n \langle n| \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|O|n\rangle$$

$$= \int d^2\alpha P(\alpha)\sum_n \langle n|\alpha\rangle\langle\alpha|O|n\rangle = \int d^2\alpha P(\alpha)\langle\alpha|O|\alpha\rangle. \tag{4.249}$$

Thus, the expectation value of products of a and $a^\dagger$ in normal order may be written

$$\langle (a^\dagger)^i a^j\rangle = \mathrm{Tr}\{\rho(a^\dagger)^i a^j\} = \int d^2\alpha P(\alpha)\langle\alpha|(a^\dagger)^i a^j|\alpha\rangle = \int d^2\alpha P(\alpha)(\alpha^*)^i\alpha^j. \tag{4.250}$$

Here, too, as in Eq. (4.204), we have an example of a quantum mechanical average evaluated by a c-number procedure. More generally,

$$\langle :O(a,a^\dagger):\rangle = \int P(\alpha)O(\alpha,\alpha^*)\,d^2\alpha, \tag{4.251}$$

that is, the expectation value of an operator in normal order is expressible as a c-number average.

We also may obtain an expression for $P(\alpha)$ when the normally ordered operator is $\chi_n(\lambda)$ as defined by Eq. (4.186b). Again, we refer to Eq. (4.249b) to write

$$\chi_n(\lambda) = \mathrm{Tr}\{\rho e^{\lambda a^\dagger}e^{-\lambda^* a}\} = \int P(\alpha)\langle\alpha|e^{\lambda a^\dagger}e^{-\lambda^* a}|\alpha\rangle\,d^2\alpha = \int P(\alpha)e^{\lambda\alpha^*-\lambda^*\alpha}\,d^2\alpha, \tag{4.252}$$

whose inverse Fourier transform, as described in Section 4.6, yields

$$P(\alpha) = \frac{1}{\pi^2}\int e^{\lambda^*\alpha-\lambda\alpha^*}\chi_n(\lambda)\,d^2\lambda. \tag{4.253}$$

Let us now derive the P-representation of the density operator at thermal equilibrium [7, 8]. We begin by computing the characteristic function $\chi_a(\lambda)$ in accordance with Eq. (4.190) where the diagonal matrix element of the thermal density operator is given by Eq. (4.242):

$$\chi_a(\lambda) = \frac{1}{\pi}\int d^2\alpha\langle\alpha|\rho_0|\alpha\rangle e^{\lambda\alpha^*-\lambda^*\alpha}$$

$$= \frac{1}{\pi(1+\langle n\rangle)}\int d^2\alpha\exp\left(-\frac{|\alpha|^2}{1+\langle n\rangle}\right)e^{\lambda\alpha^*-\lambda^*\alpha}. \tag{4.254}$$

In terms of the real and imaginary components defined in Eq. (4.191), we have

$$\chi_a(x,k) = \frac{1}{2\pi(1+\langle n\rangle)}\int\exp\left(-\frac{q^2+p^2}{2(1+\langle n\rangle)}\right)e^{i(kq-xp)}\,dq\,dp. \tag{4.255}$$

To facilitate the evaluation of the integral, note that

$$\int\exp(-at^2)e^{\pm i\omega t}\,dt = \sqrt{\frac{\pi}{a}}\exp\left(\frac{-\omega^2}{4a}\right), \tag{4.256}$$

which then leads to the simple result

$$\chi_a(\lambda) = \exp[-(1+\langle N\rangle)|\lambda|^2] = \chi_n(\lambda)\exp[-|\lambda|^2]. \tag{4.257}$$

The second equality makes reference to Eq. (4.187). Since $P(\alpha)$ is related to $\chi_n(\lambda)$ by Eq. (4.263), we have

$$P(\alpha) = \frac{1}{\pi^2}\int\exp[-\langle n\rangle|\lambda|^2]e^{\lambda^*\alpha-\lambda\alpha^*}\,d^2\lambda = \frac{1}{\pi\langle n\rangle}\exp\left[\frac{-|\alpha|^2}{\langle n\rangle}\right] \tag{4.258}$$

which again, has been evaluated with the help of Eq. (4.255). We note that if $\langle n\rangle \gg 1$, $P(\alpha)$ merges with $\langle\alpha|\rho_0|\alpha\rangle$ in Eq. (4.242). It is seen that $P(\alpha)$ for thermal or, more generally, chaotic radiation is a Gaussian function; it therefore may be interpreted as a probability distribution. For the multimode case,

$$P(\{\alpha_k\}) = \prod_k\frac{1}{\pi\langle n_k\rangle}\exp\left[-\frac{|\alpha_k|^2}{\langle n_k\rangle}\right]. \tag{4.259}$$

The insertion of Eq. (4.258) into Eq. (4.243) yields the P-representation of the thermal density operator

$$\rho_0 = \frac{1}{\pi\langle n\rangle}\int d^2\alpha\exp\left[-\frac{|\alpha|^2}{\langle n\rangle}\right]|\alpha\rangle\langle\alpha| \tag{4.260}$$

for a single mode.

Let us now compare some of the statistical properties of coherent states with the corresponding properties of photon-number states. In the latter, the number of photons n is precisely determined. Thus, with $N = a^\dagger a$,

$$\langle n\rangle_n = \langle n|N|n\rangle = n, \tag{4.261a}$$

$$\langle n^2\rangle_n = \langle n|N^2|n\rangle = n^2. \tag{4.261b}$$

We then obtain for the mean square fluctuation and the relative uncertainty

$$(\Delta n)_n^2 = \langle n^2\rangle_n - \langle n\rangle_n^2 = 0, \tag{4.262}$$

$$\eta = \frac{(\Delta n)_n}{\langle n\rangle_n} = 0. \tag{4.263}$$

In the coherent state basis, however,

$$\langle n\rangle_c = \langle\alpha|N|\alpha\rangle = |\alpha|^2, \qquad \langle n^2\rangle_c = \langle\alpha|N^2|\alpha\rangle = |\alpha|^4 + |\alpha|^2, \tag{4.264}$$

$$(\Delta n)_c^2 = \langle n^2\rangle_c - \langle n\rangle_c^2 = |\alpha|^2 = \langle n\rangle_c, \tag{4.265}$$

consistent with Poissonian statistics. Also,

$$\eta_c = \frac{(\Delta n)_c}{\langle n\rangle_c} = |\alpha|^{-1} = \langle n\rangle_c^{-1/2}. \tag{4.266}$$

We see, then, that in a coherent state the number of photons is not precise since, according to Eq. (4.265), the mean square fluctuation is equal to the average number of photons. But the relative uncertainty, η_c, approaches zero with increasing mean photon number $\langle n\rangle_c$.

Another important comparison between photon-number and coherent states is obtained when one calculates the product $\Delta Q\,\Delta P$ for each case. Referring to Eq. (4.69),

$$\begin{aligned}
&\langle n|Q|n\rangle = \langle n|P|n\rangle = 0,\\
&\langle n|Q^2|n\rangle = \frac{\hbar}{2\omega}\langle n|(a^\dagger a)^2|n\rangle = \frac{\hbar}{2\omega}(2n+1),\\
&\langle n|P^2|n\rangle = -\frac{\hbar\omega}{2}\langle n|(a^\dagger - a)^2|n\rangle = \frac{\hbar\omega}{2}(2n+1).
\end{aligned} \tag{4.267}$$

Thus,

$$(\Delta Q)_n^2 \equiv \langle n|Q^2|n\rangle - \langle n|Q|n\rangle^2 = \frac{\hbar}{2\omega}(2n+1), \tag{4.268}$$

$$\begin{aligned}
&(\Delta P)_n^2 \equiv \langle n|P^2|n\rangle - \langle n|P|n\rangle^2 = \frac{\hbar\omega}{2}(2n+1),\\
&(\Delta Q\,\Delta P)_n = \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2}.
\end{aligned} \tag{4.269}$$

The minimum uncertainty, i.e., $\Delta Q\,\Delta P = \hbar/2$, is achieved only in the vacuum state ($n = 0$). When the calculation is repeated for a coherent state we obtain, with the help of the relations in Eq. (4.143),

$$\begin{aligned}
&\langle\alpha|Q|\alpha\rangle = \sqrt{\frac{\hbar}{2\omega}}(\alpha^* + \alpha), \qquad \langle\alpha|P|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2}}(\alpha^* - \alpha),\\
&\langle\alpha|Q^2|\alpha\rangle = \frac{\hbar}{2\omega}(\alpha^{*2} + 2|\alpha|^2 + \alpha^2 + 1),\\
&\langle\alpha|P^2|\alpha\rangle = \frac{\hbar\omega}{2}(\alpha^{*2} - 2|\alpha|^2 + \alpha^2 - 1).
\end{aligned} \tag{4.270}$$

Then

$$\begin{aligned}
&(\Delta Q)_c^2 \equiv \langle\alpha|Q^2|\alpha\rangle - \langle\alpha|Q|\alpha\rangle^2 = \frac{\hbar}{2\omega},\\
&(\Delta P)_c^2 \equiv \langle\alpha|P^2|\alpha\rangle - \langle\alpha|P|\alpha\rangle^2 = \frac{\hbar\omega}{2},
\end{aligned} \tag{4.271}$$

$$(\Delta Q\,\Delta P)_c = \frac{\hbar}{2}. \tag{4.272}$$

The coherent state is therefore a *minimum uncertainty state*—a feature shared with the vacuum state in the photon-number basis.

4.9 Squeezed States

Squeezed states are among the discoveries involving laser radiation that have an important bearing on potential applications to low-noise detection systems. We begin by writing the annihilation and creation operators for a single mode in the form

$$\begin{aligned}
&a = \frac{1}{\sqrt{2}}(a_1 + ia_2), \qquad a^\dagger = \frac{1}{\sqrt{2}}(a_1 - ia_2),\\
&a_1 = \frac{1}{\sqrt{2}}(a^\dagger + a), \qquad a_2 = \frac{i}{\sqrt{2}}(a^\dagger - a),
\end{aligned} \tag{4.273}$$

in which a_1 and a_2, known as the *quadrature components* of the radiation mode, are Hermitian operators. They are related closely to the position and momentum operators of the field since, according to Eq. (4.95),

$$\begin{aligned}
&Q = \sqrt{\frac{\hbar}{2\omega}}(a^\dagger a) = \sqrt{\frac{\hbar}{\omega}}a_1,\\
&P = i\sqrt{\frac{\hbar\omega}{2}}(a^\dagger - a) = \sqrt{\hbar\omega}\,a_2.
\end{aligned} \tag{4.274}$$

The reason for referring to a_1 and a_2 as quadrature components may be seen by writing the electric field operator in terms of a_1 and a_2. For a single mode in the Heisenberg representation, we have, from Eq. (4.119),

$$\begin{aligned}
\mathbf{E}_H(\mathbf{r},t) &= K\hat{\varepsilon}i[ae^{i\theta} - a^\dagger e^{-i\theta}]\\
&= \frac{K}{\sqrt{2}}\hat{\varepsilon}i[(a_1 + ia_2)e^{i\theta} - (a_1 - ia_2)e^{-i\theta}]\\
&= -K\hat{\varepsilon}\sqrt{2}(a_2\cos\theta + a_1\sin\theta)
\end{aligned} \tag{4.275}$$