Supplementary Material for: Realizing the Harper Hamiltonian with Laser-Assisted Tunneling in Optical Lattices

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Here we derive the tunneling matrix elements for the Wannier-Stark Hamiltonian driven by a moving lattice, and obtain the Harper Hamiltonian which describes charged particles in a magnetic field.

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We start with the Wannier-Stark Hamiltonian and add a moving lattice to drive laser-assisted tunneling processes, which is described by a Raman process between states on different lattice sites. The derivation generalizes the treatment in [1, 2], and the notation utilizes that of the main paper.

\[ H = \frac{\mathbf{p}^2}{2m} + V_{\text{latt}}(\mathbf{r}) - \frac{\Delta}{a} x + \Omega \sin \left( \delta \mathbf{k} \cdot \mathbf{r} - \frac{k_x a}{2} - \omega t \right) \]  

The phase term, \(-k_x a/2\), is included for later computational convenience, and does not change the physics of the problem. Working in two dimensions and ignoring interactions, the Wannier-Stark Hamiltonian is projected onto the lowest band of a cubic lattice using a basis of localized Wannier-Stark functions in the \(x\) direction and Wannier functions in the \(y\) direction:

\[ H = \sum_{m,n} \left( -m\Delta \langle m,n| \langle m,n| - J_y \langle m,n| + 1 \rangle \langle m,n| + h.c. \right) \\
+ \sum_{m',n'} \Omega \langle m',n'| \langle m',n'| \sin \left( \delta \mathbf{k} \cdot \mathbf{r} - \frac{k_x a}{2} - \omega t \right) \langle m,n| \langle m,n| \right) \]

There are two matrix elements of interest here: the diagonal term as well as overlap of adjacent sites in the \(x\) and \(y\) directions. In general these have the form:

\[ \langle m,n| \sin \left( \delta \mathbf{k} \cdot \mathbf{r} - \frac{k_x a}{2} - \omega t \right) |m+l,n+p\rangle \]  

The phase shift of the Raman drive, \(k_x a/2\), is associated with a spatial shift for the tunneling matrix elements and a temporal shift for the onsite matrix elements making their self-consistent evaluation based on symmetry arguments easier. Using \(\mathbf{R}_{m,n} = ma \hat{x} + na \hat{y}\) for the position of the lattice sites, the relevant matrix elements can be re-written as:

\[ \langle 0,0| \sin \left( \delta \mathbf{k} \cdot (\mathbf{r} + \mathbf{R}_{m,n}) - k_x a/2 - \omega t \right) |l,p\rangle \]  

To condense notation, we define: \(\theta_{m,n} = \omega t - \delta \mathbf{k} \cdot \mathbf{R}_{m,n} = \omega t - \phi_{m,n}\), with \(\phi_{m,n} = mk_x a + nk_y a\). Expanding the \(\sin(a + b - c)\) form of the Raman operator into four terms one obtains the relevant matrix elements:

\[ \langle 0| \sin(k_y y)|p = 0\rangle = 0 \]  
\[ \langle 0| \cos(k_y y)|p = 0\rangle = \Phi_{y0}(k_y) \]  
\[ \langle 0| \sin(k_x x)|l = 0\rangle = 0 \]  
\[ \langle 0| \cos(k_x x)|l = 0\rangle = \Phi_{x0}(k_x) \]  
\[ \langle 0| \sin(k_x (x - a/2))|l = 1\rangle = \Phi_{x1}(k_x) \]  
\[ \langle 0| \cos(k_x (x - a/2))|l = 1\rangle = \Phi'_{x1}(k_x) \]

The expressions above are evaluated using maximally localized Wannier functions in the \(y\) direction and Wannier-Stark wavefunctions in the \(x\) direction. Due to the symmetric nature of the localized Wannier function [3], all matrix elements of an antisymmetric function are zero. The Wannier-Stark wavefunctions do not have definite parity as discussed in [4] so the overlap elements must be individually evaluated. In the tight-binding limit, the tunneling term
is dominated by $\Phi_{x1} \approx -2J_x \sin(k_x a/2)/\Delta$. However, at lower lattice depths $\Phi'_{x1}$ can become significant so we keep both terms.

The time dependence of the diagonal terms can be eliminated via a unitary transformation into a rotating frame given

\begin{equation}
\Omega \Phi_{y0} (\Phi_{x1} \cos \theta_{m,n} - \Phi'_{x1} \sin \theta_{m,n})
\end{equation}

In addition, the Raman-coupling induces an on-site modulation given by:

\begin{equation}
- \Omega \Phi_{x0} \Phi_{y0} \sin(\theta_{m,n} + k_x a/2)
\end{equation}

Given the above form for the on- and offdiagonal terms, we arrive at an effective Hamiltonian:

\begin{equation}
H = \sum_{m,n} \left[ -m \Delta - \Omega \Phi_{x0} \Phi_{y0} \sin(\theta_{m,n} + k_x a/2) \right]|m,n\rangle\langle m,n| + \ldots
\end{equation}

\begin{equation}
\ldots + \Omega \Phi_{y0} (\Phi_{x1} \cos \theta_{m,n} - \Phi'_{x1} \sin \theta_{m,n})|m+1,n\rangle\langle m,n| - J_y|m,n+1\rangle\langle m,n| + h.c.\right]
\end{equation}

The time dependence of the diagonal terms can be eliminated via a unitary transformation into a rotating frame given by:

\begin{equation}
U = \exp \left[ i \sum_{m,n} (m \omega t - \frac{\Omega \Phi_{x0} \Phi_{y0}}{\hbar \omega} \cos \left( \theta_{m,n} + \frac{k_x a}{2} \right)) |m,n\rangle\langle m,n| \right] = \sum_{m,n} e^{i \Lambda_{m,n}} |m,n\rangle\langle m,n|
\end{equation}

where $\Lambda_{m,n} = m \omega t - \frac{\Omega \Phi_{x0} \Phi_{y0}}{\hbar \omega} \cos(\theta_{m,n} + k_x a/2)$. In this frame the Hamiltonian becomes $H' = U^\dagger H U - i\hbar U^\dagger (dU/dt)$. For the case of resonant drive where $\hbar \omega = \Delta$, the diagonal terms are zero, leaving only off-diagonal elements for tunneling in the $x$ and $y$ directions. Considering tunneling in the $x$ direction first and using the full expression for $\theta_{m,n}$, the exponential factor is:

\begin{equation}
e^{-i(\Lambda_{m+1,n} - \Lambda_{m,n})} = \exp \left[ -i \left( \omega t - \frac{\Omega \Phi_{x0} \Phi_{y0}}{\hbar \omega} \left( \cos \left( \omega t + \frac{k_x a}{2} - \phi_{m+1,n} \right) - \cos \left( \omega t + \frac{k_x a}{2} - \phi_{m,n} \right) \right) \right) \right]
\end{equation}

\begin{equation}
= \exp \left[ -i \left( \omega t - \frac{2 \Omega \Phi_{x0} \Phi_{y0}}{\hbar \omega} \left( \sin \left( \frac{k_x a}{2} \right) \sin \left( \omega t - \phi_{m,n} \right) \right) \right) \right]
\end{equation}

For tunneling in the $y$ direction, the transformation into the rotating frame adds the exponential factor:

\begin{equation}
e^{-i(\Lambda_{m,n+1} - \Lambda_{m,n})} = \exp \left[ i \frac{\Omega \Phi_{x0} \Phi_{y0}}{\hbar \omega} \left( \cos \left( \omega t + \frac{k_x a}{2} - \phi_{m,n+1} \right) - \cos \left( \omega t + \frac{k_x a}{2} - \phi_{m,n} \right) \right) \right]
\end{equation}

\begin{equation}
= \exp \left[ i \frac{2 \Omega \Phi_{x0} \Phi_{y0}}{\hbar \omega} \left( \sin \left( \frac{k_y a}{2} \right) \cos \left( \omega t + \frac{(k_x - k_y) a}{2} - \phi_{m,n} \right) \right) \right]
\end{equation}

If we define the quantity $\Gamma_x(y) = \frac{2 \Omega \Phi_{x0} \Phi_{y0}}{\hbar \omega} \sin(k_x a/2)$ and use the Jacobi-Anger identity, $e^{ix \sin(\theta)} = \sum_r J_r(x)e^{ir\theta}$, both above expressions can be simplified to:

\begin{equation}
e^{-i(\Lambda_{m+1,n} - \Lambda_{m,n})} = e^{-i\omega t} \sum_{r=-\infty}^{\infty} J_r(\Gamma_x) e^{ir(\omega t - \phi_{m,n})}
\end{equation}

\begin{equation}
e^{-i(\Lambda_{m,n+1} - \Lambda_{m,n})} \sum_{r=-\infty}^{\infty} J_r(\Gamma_y) e^{ir(\omega t + (k_x - k_y) a/2 - \phi_{m,n})}
\end{equation}

for $x$ and $y$ tunneling, respectively. Here, $J_r(\Gamma)$ are the Bessel functions of the first kind. Now the time-dependent tunneling amplitudes $K(t) = \langle m+1,n | H' | m,n \rangle$ and $J(t) = \langle m,n+1 | H' | m,n \rangle$ in the rotating frame are given by:

\begin{equation}
K(t) = \Omega \Phi_{y0} (\Phi_{x1} \cos(\omega t - \phi_{m,n}) - \Phi'_{x1} \sin(\omega t - \phi_{m,n})) e^{-i\omega t} \sum_r J_r(\Gamma_x) e^{ir(\omega t - \phi_{m,n})}
\end{equation}

\begin{equation}
J(t) = -J_y \sum_r J_r(\Gamma_y) e^{ir(\omega t + (k_x - k_y) a/2 - \phi_{m,n})}
\end{equation}

So far we have transformed Eq. (12) exactly. Time averaging over a period of $\tau \sim 1/\Delta$ gives tunneling rates for the effective Hamiltonian, $H_{eff} = \langle H' \rangle$:

\begin{equation}
K_{eff} = \frac{\Omega \Phi_{y0}}{2} e^{-i\phi_{m,n}} \left[ \Phi_{x1} \left( J_0(\Gamma_x) + J_2(\Gamma_x) \right) + i \Phi'_{x1} \left( J_0(\Gamma_x) - J_2(\Gamma_x) \right) \right] = Ke^{-i\phi_{m,n}}
\end{equation}

\begin{equation}
J_{eff} = -J_y J_0(\Gamma_y) = J
\end{equation}
Using the Bessel function identities \( \frac{1}{2}(J_0(x) + J_2(x)) = J_1(x)/x \) and \( \frac{1}{2}(J_0(x) - J_2(x)) = \frac{dJ_1(x)}{dx} \), the above expression for \( K \) simplifies to:

\[
K = \Omega \Phi_{y0} \Phi_x e^{-i\phi_{m,n}} \left( \Phi_{x1} \frac{J_1(\Gamma_x)}{\Gamma_x} + i\Phi'_{x1} \frac{dJ_1(\Gamma_x)}{d\Gamma_x} \right)
\]  

(22)

These constitute the coefficients of an effective Hamiltonian in the rotating frame exactly analogous to the Harper Hamiltonian:

\[
H_{\text{eff}} = \sum_{\langle m,n \rangle} (K e^{-i\phi_{m,n}} |m+1, n\rangle \langle m, n| + J|m, n+1\rangle \langle m, n| + h.c.)
\]  

(23)

Within the tight-binding model, where \( \Phi_{x1} \approx -2J_x \sin(k_xa/2)/\Delta \gg \Phi'_{x1} \) and \( \Phi_{y0} = \Phi_{x0} \approx 1 \), the expression for the tunneling amplitude in the \( x \) direction becomes:

\[
K \approx -J_x J_1 \left( \frac{2\Omega}{\Delta} \sin \left( \frac{k_x a}{2} \right) \right) = -J_x J_1 \left( \frac{2\Omega}{\Delta} \right)
\]  

(24)

where the last equality is for the specific case where \( k_x a = k_y a = \pi \), as in our experiment.

Furthermore, in the limit of low Raman lattice depths, \( \Gamma_x \ll 1 \), where \( J_0(\Gamma_x) \approx 1 \gg J_2(\Gamma_x) \), Eq. 20 reduces to the perturbative expression for laser-assisted tunneling given in the main text:

\[
K_{\text{eff}} = \frac{\Omega \Phi_{y0}}{2} e^{-i\phi_{m,n}} \left( \Phi_{x1} + i\Phi'_{x1} \right) = i\frac{\Omega}{2} e^{-i\phi_{m,n}} \langle m = 0 | e^{-ik_x x} | m = 1 \rangle \langle n = 0 | \cos(k_y y) | n = 0 \rangle
\]  

(25)

\[
= i\frac{\Omega}{2} e^{-i\phi_{m,n}} \langle 0, 0 | e^{-i\delta k \cdot r} | 1, 0 \rangle
\]  

(26)