CIRCUIT IMPLEMENTATIONS OF SOLITON SYSTEMS

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Recently, a large class of nonlinear systems which possess soliton solutions has been discovered for which exact analytic solutions can be found. Solitons are eigenfunctions of these systems which satisfy a form of superposition and display rich signal dynamics as they interact. In this paper, we view solitons as signals and consider exploiting these systems as specialized signal processors which are naturally suited to a number of complex signal processing tasks. New circuit models are presented for two soliton systems, the Toda lattice and the discrete-KdV equations. These analog circuits can generate and process soliton signals and can be used as multiplexers and demultiplexers in a number of potential soliton-based wireless communication applications discussed in [Singer et al.]. A hardware implementation of the Toda lattice circuit is presented, along with a detailed analysis of the dynamics of the system in the presence of additive Gaussian noise. This circuit model appears to be the first such circuit sufficiently accurate to demonstrate true overtaking soliton collisions with a small number of nodes. The discrete-KdV equation, which was largely ignored for having no prior electrical or mechanical analog, provides a convenient means for processing discrete-time soliton signals.

1. Introduction

Many traditional signal processing applications rely on models that are inherently linear and time-invariant (LTI). Much of the success of such methods can be attributed to their being mathematically tractable, often leading to efficient signal representations and fast algorithms. Linear techniques have also proven effective for modeling a variety of signals of practical interest such as speech or financial time-series and systems of interest such as telephone or radio broadcast channels. However, we increasingly turn to nonlinear models to capture some of the more salient behavior of physical or natural systems that cannot be expressed by linear means, such as threshold phenomena, amplitude-dependence, or chaotic behavior. Nonlinear systems also hold the potential to produce more efficient algorithms or models for a variety of signal processing and communication problems where linear techniques are suboptimal.

The class of nonlinear systems that support soliton solutions appears to be of particular interest to explore for signal synthesis and analysis. Solitons have been observed in a variety of natural phenomena from water and plasma waves [Infeld & Rowlands, 1990; Scott et al., 1973] to crystal lattice vibrations [Fermi et al., 1965] and energy transport.

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In proteins [Infeld & Rowlands, 1990]. Recently, solitons have become of significant interest for optical telecommunications, where optical pulses have been shown to propagate as solitons in appropriately tailored nonlinear media for tremendous distances without significant loss or dispersion [Haus, 1993].

In this paper, we view solitons from a very different perspective. Rather than focusing on the propagation of solitons through nonlinear media, we consider the implementation of circuits which can generate and process signals for transmission over ideal linear channels. In this context, we then consider these nonlinear circuits as specialized signal processors for exploiting soliton signals.

Systems that support solitons share many of the properties that make LTI systems attractive from an engineering standpoint. Although nonlinear, these systems are analytically solvable through a technique called inverse scattering, which is analogous to the Fourier transform for linear systems [Ablowitz & Clarkson, 1991]. Solitons are eigenfunctions of these systems and satisfy a nonlinear form of superposition. We can therefore decompose complex solutions in terms of a class of signals with simple dynamical structure.

In this paper, we examine the properties of solitons as signals and propose and investigate circuits that can be used to generate them. Section 2 provides an introduction to some of the properties of soliton systems paying particular attention to two such systems: The Toda lattice equation and the discrete Korteweg deVries (KdV) equation. In Sec. 3, we develop new circuit models for these two soliton systems. The first is a diode ladder implementation of the Toda lattice equation which appears to be the first Toda lattice circuit implementation sufficiently accurate to display true soliton behavior over a small number of nodes. We also develop a lattice-circuit implementation of the discrete-KdV equation, which is important for processing discrete-time soliton signals. These circuit models form the basis for a communication paradigm in which multiple signals can be multiplexed onto soliton carriers using such circuits as tuned modulators and demodulators. This paradigm is developed in a related paper [Singer et al.]. In order to utilize soliton systems in any practical context, accurate models are needed for the effects of disturbances on the dynamics of these systems. In Sec. 4, we analyze the effects of small amplitude noise on the dynamics of solitons in the Toda lattice and characterize the statistics of the noise as it propagates through the system.

2. Soliton Systems

An important class of solutions to certain nonlinear evolution equations are traveling wave solutions that propagate with constant shape and velocity; these are referred to as “solitary waves”. Specifically, a solitary wave solution with temporal and spatial variables, \( t \) and \( n \), is a traveling wave of the form, \( u(n, t) = f(n - ct) = f(z) \), where \( c \) is a fixed constant, and the energy of \( f(z) \) is localized in \( z \).

There are many physical systems that support solitary wave solutions [Ablowitz & Clarkson, 1991; Haus, 1993; Scott, 1970]. In this paper we focus primarily on two, referred to as the Toda lattice [Toda, 1989] and discrete-KdV [Ablowitz & Clarkson, 1991; Toda, 1981] equations.

2.1. The Toda lattice

The Toda lattice equations describe the displacements of an infinite chain of masses connected with nonlinear springs, as illustrated in Fig. 1. Each of the springs satisfies the nonlinear force law

\[
f_n = a(e^{-b(y_n-y_{n-1})} - 1),
\]

where \( f_n \) is the force on the spring between masses with displacements \( y_n \) and \( y_{n-1} \) from their rest positions. The equations of motion for the lattice are given by

\[
m \frac{d^2}{dt^2} y_n(t) = a(e^{-b(y_n(t)-y_{n-1}(t))} - e^{-b(y_{n+1}(t)-y_n(t))}),
\]

where \( y_n(t) \) is the displacement of the \( n \)th mass from its rest position, \( m \) is the mass, and \( a \) and \( b \) are constants.

Equation (2) with \( m = a = b = 1 \), admits solitary wave solutions of the form [Toda 1981]

\[
f_n(t) = \beta^2 \text{sech}^2(\sinh^{-1}(\beta)n - \beta t),
\]

which propagate as compressional waves stored as forces in the nonlinear springs. A single right-traveling wave \( f_n(t) \) is shown in Fig. 2. The bottom trace in the figure corresponds to the force in the spring between masses “zero” and “one” of an infinite-length Toda chain. This compressional wave is localized in time, and propagates along
Fig. 1. The Toda lattice.

Fig. 2. A propagating wave solution to the Toda lattice equations. Each trace corresponds to the force $f_n(t)$ stored in the spring between mass $n$ and $n-1$.

![Image of Toda lattice]

In the spring between masses the force in the spring between mass $n$ and $n-1$ is given by Eq. (3). Each trace in the figure corresponds to the force in the spring between masses at the associated indices. Note that when the larger soliton catches up to the smaller soliton as viewed on the fifteenth node, the combined amplitude of the two solitons is actually less than would be expected for a linear system, which would display a linear superposition of the two amplitudes. Also, the signal shape changes significantly during this nonlinear interaction.

If a solution to the equation is composed of solitary waves with different amplitudes, then collisions between the solitary waves are possible. The term “soliton” refers to such solitary wave solutions which retain their identity upon collision with other solitary waves. Figure 3 illustrates soliton behavior in the Toda lattice for two solutions of the form of Eq. (3). Each trace in the figure corresponds to the force in the spring between masses at the associated indices. Note that when the larger soliton catches up to the smaller soliton as viewed on the fifteenth node, the combined amplitude of the two solitons is actually less than would be expected for a linear system, which would display a linear superposition of the two amplitudes. Also, the signal shape changes significantly during this nonlinear interaction.

Fig. 3. Two solitary wave solutions to the Toda lattice.

An analytic expression for the two-soliton solution for $\beta_1 > \beta_2 > 0$ is given by [Hirota & Suzuki, 1973]

$$f_n(t) = \frac{\beta_1^2 \text{sech}^2(\eta_1) + \beta_2^2 \text{sech}^2(\eta_2) + A \text{sech}^2(\eta_1) \text{sech}^2(\eta_2)}{\left( \cosh \left( \frac{\phi}{2} \right) + \sinh \left( \frac{\phi}{2} \right) \tanh(\eta_1) \tanh(\eta_2) \right)},$$

where

$$A = \sinh \left( \frac{\phi}{2} \right) \left( (\beta_1^2 + \beta_2^2) \sinh \left( \frac{\phi}{2} \right) + 2\beta_1\beta_2 \cosh \left( \frac{\phi}{2} \right) \right),$$

and

$$\phi = \ln \left( \frac{\sinh \left( \frac{p_1 - p_2}{2} \right)}{\sinh \left( \frac{p_1 + p_2}{2} \right)} \right),$$

and $\beta_i = \sinh(p_i)$, and $\eta_i = p_i n - \beta_i(t - \delta_i)$.

Although Eq. (4) appears rather complex, Fig. 3 illustrates that for large separations, $|\delta_1 - \delta_2|$, $f_n(t)$ essentially reduces to the linear superposition of two solitons with parameters $\beta_1$ and $\beta_2$. As the relative separation decreases, the multiplicative cross-term becomes significant and the solitons interact nonlinearly. This asymptotic behavior can also be evidenced analytically,

$$f_n(t) = \beta_1^2 \text{sech}^2 \left( p_1 n - \beta_1(t - \delta_1) \pm \frac{\phi}{2} \right) + \beta_2^2 \text{sech}^2 \left( p_2 n - \beta_2(t - \delta_2) \mp \frac{\phi}{2} \right),$$

$$t \to \pm \infty,$$
where each component soliton experiences a net displacement $\phi$ from the nonlinear interaction.

The Toda lattice also admits periodic solutions which can be written in terms of the Jacobian elliptic functions $dn(\cdot)$ and $sn(\cdot)$. These solutions can be expressed

$$f_n(t) = (2K\nu)^2 \left\{ \text{dn} \left[ 2 \left( \frac{n}{\lambda} \pm \nu t \right) K \right] - \frac{E}{K} \right\}, \quad (8)$$

for wavelength $\lambda$ and frequency $\nu$, with

$$2K\nu = \left( \text{sn}^{-2} \left( \frac{2K}{\lambda} \right) - 1 \right)^{-1/2}, \quad (9)$$

where $K$ and $E$ are complete elliptic integrals of the first and second kind, respectively [Toda, 1981].

When written in terms of the spring forces, the Toda lattice equations become

$$\frac{d^2}{dt^2} \ln(1 + f_n(t)) = (f_{n+1}(t) - 2f_n(t) + f_{n-1}(t)). \quad (10)$$

If the substitution

$$f_n(t) = \frac{d^2}{dt^2} \ln \psi_n(t) \quad (11)$$

is made into Eq. (10), then the lattice equations become

$$\ddot{\psi}_n - \psi_n \dot{\psi}_n = \psi_n^2 - \psi_{n-1}\psi_{n+1}. \quad (12)$$

In view of the Teager energy operator introduced by Kaiser [1990], the left-hand side of Eq. (12) is the Teager instantaneous-time energy at the node $n$, and the right-hand side is the Teager instantaneous-space energy at time $t$. In this form, we may view solutions to Eq. (12) as propagating waveforms that have equal Teager energy as calculated in time and in space, a relationship also observed by Kaiser.

### 2.2. Discrete-KdV

The discrete-KdV (dKdV) equation, sometimes referred to as the nonlinear ladder equations [Ablowitz & Clarkson, 1991] or the KM system (Kac and van Moerbeke) [Toda, 1981], is governed by the equations,

$$\frac{d}{dt} u_n(t) = e^{u_{n-1}(t)} - e^{u_{n+1}(t)}. \quad (13)$$

These equations are first-order in time, which makes the dynamics less complex than the Toda lattice to which the dKdV equation is closely related. Although much of the theory for this nonlinear system has been developed in association with the theory for the Toda lattice, the discrete-KdV equations are generally ignored since there is no clear physical analog of these equations. However, there is special relationship known as a Bäcklund transformation which provides a connection between this system and the Toda lattice [Kac & van Moerbeke, 1975a, 1975b; Toda, 1981].

Following [Kac & van Moerbeke, 1975a, 1975b; Toda, 1981], let

$$u_n(t) \rightarrow -R_n(t), \quad t \rightarrow -t, \quad (14)$$

which transforms (13) to

$$\frac{d}{dt} R_n(t) = e^{-R_{n-1}(t)} - e^{-R_{n+1}(t)}. \quad (15)$$

Letting $q_n = R_n + R_{n+1}$, then $q_n$ satisfies

$$\frac{d^2}{dt^2} q_n(t) = 2e^{-q_n(t)} - e^{-q_{n-2}(t)} - e^{q_{n+2}(t)}. \quad (16)$$

Taking every other term, i.e. $q_{1,n} = q_{2,n}$, $q_{2,n} = q_{2n+1}$, we have

$$\frac{d^2}{dt^2} q_{i,n}(t) = 2e^{-q_{i,n}(t)} - e^{-q_{i,n-1}(t)} - e^{q_{i,n+1}(t)}, \quad i = 1, 2. \quad (17)$$

Setting $q_{i,n} = -\ln(1 + f_{i,n})$, yields

$$\frac{d^2}{dt^2} \ln(1 + f_{i,n}(t)) = f_{i,n-1}(t) - 2f_{i,n}(t) + f_{i,n+1}(t), \quad i = 1, 2, \quad (18)$$

which are each a Toda lattice equation. The physical interpretation of this transformation is the following: If $f_{1,n}$ and $f_{2,n}$ are each defined within a different Toda lattice, then $R_n$ defined by $R_{2n} = f_{1,n} - f_{2,n}$, $R_{2n+1} = f_{2,n+1} - f_{1,n}$ satisfies the discrete-KdV equation in the form (15). This process is illustrated in Fig. 4.

The single soliton solution for $N_n = e^{u_n}$ is given by [Manakov, 1975]

$$N_n(t) = \frac{\cosh(\eta(n-x_0(t)-2))\cosh(\eta(n-x_0(t)+1))}{\cosh(\eta(n-x_0(t)-1))\cosh(\eta(n-x_0(t)))}. \quad (19)$$

where

$$x_0(t) = x_0(0) + \frac{\sinh(2\eta)}{\eta} t. \quad (20)$$
behavior. In [Singer et al.], we consider exploiting these circuits in a communication context.

### 3.1. Toda circuit model of Hirota and Suzuki

Motivated by the work of Toda on the exponential lattice, the nonlinear LC ladder network shown in Fig. 5, using linear inductors and nonlinear capacitors, was given by Hirota and Suzuki [1973]. Rather than using a direct analog of the Toda lattice, the authors derived the functional form of the nonlinear capacitance required such that the capacitor voltages in the LC line would have the same dynamics as the forces in the nonlinear springs. The resulting network equations are given by

\[
\frac{d^2}{dt^2} \ln \left(1 + \frac{V_n(t)}{V_0}\right) = \frac{1}{LC_0V_0} (V_{n-1}(t) - 2V_n(t) + V_{n+1}(t)),
\]

which are equivalent to the Toda lattice equations for the forces on the nonlinear springs given in Eq. (10). This amounts to an implicit mapping from force to voltage, \(f_n(t) \rightarrow V_n(t)\). The capacitance required in the nonlinear LC ladder is of the form

\[
C(V) = \frac{C_0V_0}{V_0 + V},
\]

where \(V_0\) and \(C_0\) are constants representing a bias voltage and a nominal capacitance, respectively. In their implementation, varactor diodes with nonlinear capacitance

\[
C(V) \approx 27(V - V_b)^{-0.48} \text{ pF},
\]

where \(V_b\) is a bias voltage, were used to approximate the required capacitance of Eq. (23).

Although the varactor diode capacitance can be biased to yield a match for small voltages, for larger voltages, the deviations from the ideal capacitance become apparent. Moreover, as the length of the lattice increases, the effects on any propagating solitons accumulate. The net result is that interaction between solitary waves of appreciable amplitude will not result in soliton collisions; rather such a collision will also produce a nontrivial amount of ripple [Hirota & Suzuki, 1973]. Also, since the circuit is only accurate for small voltages, where the velocity difference between solitons is small, large numbers of nodes are required to bring about collisions of solitons propagating at different velocities in the same direction.
After publication of their circuit [Hirota & Suzuki, 1973] and subsequent publication of modulation experiments using the circuit [Suzuki et al., 1973a, 1973b], several papers appeared in the literature on a variety of related topics. In [Kolosick et al., 1974] a similar nonlinear network is analyzed. Islam et al. [1987] studied the effects of dissipation on the propagation of individual solitons as well as the interaction of pairs of solitons. It was found that dissipative effects led to a decrease in amplitude and an increase in the width of solitons as they propagate through the lattice. These findings are in agreement with the numerical work of Okada et al. [1990], whose studies showed similar effects due to parameter fluctuation in the periodic Toda lattice. Ballantyne et al. [1993] observed the Jacobian elliptic function solutions in a periodic version of the nonlinear LC line. Toda also demonstrated properties of the nonlinear line and illustrated the existence of modulated solitons, by relating the lattice to the nonlinear Schrödinger equation in [Toda, 1989]. Finally, Cho et al. developed a similar nonlinear network as an equivalent circuit model for the propagation of nonlinear surface acoustic waves in thin-bar and broad-plate vibrations. They also have shown that the nonlinear LC network is an accurate model for a metallic grating waveguide and use this circuit model to explain certain nonlinearities observed in SAW devices including the generation of acoustic phase-conjugate waves [Cho et al., 1993].

### 3.2. Diode ladder circuit model for Toda lattice

Although the nonlinear ladder network realizations of the Toda lattice retain many of the properties of the ideal lattice, as suggested in Sec. 3.1, the dynamics of these circuits are limited to a small range of voltages and therefore their applicability is inherently limited. In this section, we present a new circuit model that more accurately represents the Toda lattice and is a direct electrical analog of the nonlinear spring mass system. If voltages $v_{n-1}$ and $v_n$ are applied to the terminals of a junction diode, then the current through the device can be

\[ i_n = I_s (v_{n-1} - v_n) / v_t - 1, \]

where $I_s$ is the saturation current and $v_t$ is the thermal voltage. If we place the diodes in a ladder configuration as shown in Fig. 6, then the current through the $n$th shunt impedance is given by

\[ i_n - i_{n+1} = I_s (v_{n-1} - v_n) / v_t - e^{(v_n - v_{n+1}) / v_t}. \]

In analogy to Eq. (2), we see that if the shunt impedance has the voltage–current relation

\[ \frac{d^2 v_n(t)}{dt^2} = \alpha (i_n(t) - i_{n+1}(t)), \]

then the governing equations for the network become

\[ \frac{d^2 v_n(t)}{dt^2} = \alpha I_s (e^{(v_{n-1} - v_n(t))/v_t} - e^{(v_n(t) - v_{n+1}(t))/v_t}), \]

or equivalently,

\[ \frac{d^2}{dt^2} \ln \left(1 + \frac{i_n(t)}{I_s}\right) = \frac{\alpha}{v_t} (i_{n-1}(t) - 2i_n(t) + i_{n+1}(t)), \]

where $i_1(t) = i_{in}(t)$. These are equivalent to the Toda lattice equations with $a/m = \alpha I_s$ and $b = 1/v_t$. The required shunt impedance is often referred to as a double capacitor, which can be realized using ideal operational amplifiers in the gyrator circuit shown in Fig. 7, yielding the required impedance of

\[ Z_n = \alpha / s^2 = R_3 / R_1 R_2 C^2 s^2 \]

[Horowitz & Hill, 1989; Siebert, 1986].

When $i_{in}(t)$ in Fig. 6 is of the form

\[ i_{in}(t) = I_s \Omega^2 \text{sech}^2(\gamma t), \]

\[ \gamma = \Omega \sqrt{I_s / v_t}, \]

a single soliton is induced in the line resulting in

\[ i_n(t) = I_s \Omega^2 \text{sech}^2(pn - \gamma t), \]
where $\Omega = \sinh(p)$. Note that the saturation current $I_s$ may be absorbed into the parameter $\Omega$, yielding

$$i_n(t) = \beta^2 \text{sech}^2(pn - \beta \tau),$$

(32)

where $\beta = \sqrt{I_s \sinh(p)}$, and $\tau = t\sqrt{\alpha/v_t}$. Since $I_s$ is generally on the order of picoamps, the operating range of the circuit can be on the order of milliamps over a wide range of values of the soliton wavenumber $p$. As a result, the diode ladder circuit model is accurate over a significantly larger range of soliton wavenumbers than is the LC circuit of Hirota and Suzuki.

Solitons of the form of Eq. (32) are solutions of the infinite-length Toda lattice equations. In practice, a finite-length lattice can be constructed to yield soliton solutions if the diode ladder circuit can be appropriately terminated to limit reflections. As a starting point, we consider the termination that would yield no reflections for the small signal model. This can be obtained from the impedance of the line when the diodes are replaced with their equivalent linearized resistance $R_{eq} = v_t/i_d$, where $i_d$ is the current in the linearized diode. This results in an impedance

$$Z_{in} = \frac{R_{eq}}{2} \pm \sqrt{\frac{R_{eq}^2}{4} + \frac{R_{eq} \alpha}{s^2}},$$

(33)

which can be approximated at high frequencies by a resistance and at lower frequencies by a resistance in series with a capacitance.

The diode lattice has been simulated in the circuit simulation package HSPICE [1992] and also implemented using standard circuit components. In the simulation, we used component models representing the circuit components used in the implementation. In the following subsections, we describe issues and results associated with the simulation and with the implementation.

### 3.3. Circuit simulation

The diode ladder has been simulated using realistic component models in the circuit simulation package HSPICE [1992]. The diodes used are model 1n4148 with a saturation current of $I_s \approx 0.01$ pA, and the operational amplifiers are model LT1028A. Setting the operation range of the circuit to produce solitons on the order of 10 mA yields a value of $p \approx 14$. To fix the time scale of the circuit, we set the pulse width of a soliton to approximately 5 $\mu$s, which leads to

$$\sinh(p) \sqrt{\frac{I_s \alpha}{v_t}} \approx \frac{1}{5 \mu s},$$

(34)

or $\alpha \approx 10^{11}$. The resistor values in the double capacitor circuits can now be chosen to prevent saturation of the operational amplifiers. By calculating the transfer function from the driving point of the double capacitor to each of the operational amplifier output voltages, we obtain

$$G_1 = \frac{R_2 + R_3}{R_3},$$

(35)

$$G_2 = 1 + \frac{R_2 + 2R_3}{R_3} R_1 C_s,$$

(36)

where $G_1$ and $G_2$ are the transfer characteristics from the voltage $v_n$ to the outputs of the top and bottom amplifiers of the gyrator circuit, respectively. In order to select a valid set of resistor values, the range of voltages at the top of the double capacitor is needed. For a single soliton solution, the closed-form solution for the voltage is

$$v_n(t) = v_t \ln \left\{ \cosh \left( p(n) - \beta t \sqrt{\frac{\alpha}{v_t}} \right) \right\}$$

$$- v_t \ln \left\{ \cosh \left( p(n + 1) - \beta t \sqrt{\frac{\alpha}{v_t}} \right) \right\}$$

$$+ \text{const}.$$

(37)
The limiting voltage in Eq. (37) is given by

\[ \lim_{t \to \infty} v_n(t) = v_tp + \text{const}, \tag{38} \]

and

\[ \lim_{t \to -\infty} v_n(t) = -v_tp + \text{const}. \tag{39} \]

Selecting the constants such that \( v_n(-\infty) = 0 \), gives

\[ \lim_{t \to \infty} v_n(t) = 2v_tp. \tag{40} \]

For \( p \approx 14 \), this leads to a final voltage amplitude on the order of \( v_n \approx 0.75 \) volts. For each soliton that passes through a given node, the voltage on the double capacitor will increase by \( 2v_tp \). A reasonable balance between signal strength and circuit linearity can be obtained by setting \( R_2 \approx R_3 \), and \( R_1 \ll 1/C \) which can be met by selecting \( R_1 = R_2 = R_3 = 1 \, \text{k}\Omega \) and \( C = 0.01 \, \mu\text{F} \). These values permit soliton pulse widths of about 5 \( \mu\text{s} \) with amplitudes of about 10 mA and with voltages at the amplifier outputs within the double capacitors on the order of 1 volt. Shown in Fig. 8 is an HSPICE simulation with two solitons propagating down a length 10 Toda chain.

A significant difference between soliton solutions to this circuit and those of the nonlinear LC line lies in the scale of operation. Due to biasing constraints for the LC line, solitons were generally restricted to a small range of wavenumbers in the neighborhood of \( p \approx 1 \). Over this range, the propagation velocity of the solitons, which is proportional to \( \sinh(p)/p \) does not vary greatly between solitons of different wavenumbers. This led to the use of chains with hundreds of nodes in order to induce overtaking soliton collisions. The diode ladder circuit, however, can operate in the range \( p \approx 14 \) for solitons with amplitudes in the milliamp range. Due to the exponential nature of the \( \sinh(\cdot) \) function, the velocities of solitons with slightly different amplitudes for currents in the milliamp range yield significantly different velocities. This enables soliton collisions to take place with far fewer nodes than with the nonlinear LC network.

As illustrated in the bottom trace of Fig. 8, a soliton can be generated by driving the circuit with a square pulse of approximately the same area as the desired soliton. As seen on the third node in the lattice, once the soliton is excited, the nonsoliton components are quickly stripped away. For the example shown in the figure, a signal containing a small pulse followed by a larger pulse is used to drive the circuit giving rise to a small amplitude soliton followed by a larger amplitude soliton. This property has been demonstrated experimentally by others for a number of soliton systems, c.f. [Haus, 1993] for the nonlinear Schrödinger equation and [Hirota & Suzuki, 1973] for the Toda lattice. It has been shown theoretically for KdV, c.f. [Ablowitz & Clarkson, 1991; Drazin & Johnson, 1989], that

![Fig. 8. HSPICE simulation of the evolution of a two-soliton signal through the diode lattice. Each horizontal trace shows the current through one of the diodes 1, 3, 4 and 5.](image-url)
practically any smooth, localized disturbance of the proper area will result in a soliton with that area, if such a solution exists.

Note that as the faster soliton overtakes the slower as viewed on the fourth node in Fig. 8, the joint signal amplitude is significantly less than the sum of the individual amplitudes. Also, the signal shape changes significantly during the nonlinear interaction. These two effects will impact both the energy of multisoliton signals and the ability to recover their signal parameters as described in [Singer et al.].

3.4. Circuit implementation

To perform real-time experimentation and to verify the operation of the model, a diode ladder circuit with twenty nodes has been implemented with standard circuit components. Real-time implementation also enables rapid testing of soliton processing techniques and enables measurements of actual circuit noise levels. Such noise measurements permit experimental verification of some of the theoretical results concerning system noise in Sec. 4.

In the construction of the circuit, there were several practical matters to be dealt with. First, the diode ladder is driven by a current source. In our implementation, the precision bipolarity current source shown in Fig. 9 taken from [Horowitz & Hill, 1989] was used. When implemented with the low noise LT1028A operational amplifiers, this circuit provides a reliable, accurate current source with low leakage. In practice, leakage current turns out to be a problem, since the double capacitor circuits are marginally stable. The node voltages are double integrators of their current and therefore any excess current will lead to large deviations in the node voltages and will corrupt soliton propagation.

In addition to the voltage deviations from leakage current, each soliton that passes through a node on the ladder contributes a net voltage increase of $2v_p$ or approximately 1 volt. Therefore a signal containing three solitons will leave the node with a net voltage increase of nearly 3 volts. If several such signals are processed by this circuit, the operational amplifiers in the double capacitors will eventually saturate. This problem can be overcome by resetting the node voltages using analog switches as shown in Fig. 10 after each processed signal.

As indicated previously, the diode ladder equations are given for an infinite ladder of identical nodes. Any practical finite implementation must be terminated in a manner to limit end-effects. By driving the lattice at one end with a current source, forward traveling solitons can be easily induced into the network. At the other end, we used an impedance which approximates the linearized impedance of the infinite network (33). For typical component values, $\alpha \approx 10^{11}$. If $R_{eq}$ is taken to be $v_t/25$ mA = 1, then for frequencies below 1 MHz, a load impedance consisting of a 1 $\Omega$ resistor and a 0.3 $\mu$F capacitor yield a good approximation with negligible reflections in practice.

Finally, since the solitons are present in the diode ladder circuit as current waveforms, there must be an adequate means of measuring the
current through the diodes without significantly affecting the dynamics of the circuit. This can be accomplished by placing a small resistance in series with each diode in the lattice as shown in Fig. 10. The current through the diodes can then be observed by measuring the voltage drop across each of the resistors with a differential amplifier.

A two-soliton signal generated by an implementation of the circuit is shown on the oscilloscope traces in Fig. 11. The bottom trace in the figure corresponds to the input current to the circuit, and the remaining traces, from bottom to top, show the current through the second, third and fourth diodes in the lattice. Another example of the circuit output is shown in Fig. 12. For this example, a simple waveform consisting of three component soliton signals, periodically repeated, was used to drive the diode ladder circuit. A digital oscilloscope was used to sample the real-time circuit waveforms, which were then transferred to a computer, and then plotted online. The time axis of the figure is such that $t = 0$ corresponds to the beginning of a period.

The largest amplitude soliton in the figure measures 17 mA, with a pulse width of 82 µs at the first node. As measured on the fifteenth node, the amplitude is 14 mA with pulse width of 86 µs. This decay in the soliton amplitude is on the order of 1% per node and may have several causes in addition to deviations of the circuit components from their idealized models. Specifically, as stated in [Islam et al., 1987], dissipative effects in the lattice are contrary to the conservative nature of the Toda lattice, and will necessarily lead to energy loss. A preliminary analysis of the linearized model, gives an indication that this series resistance in combination with a slight diode leakage current of 0.25 mA would lead to a reduction in energy of 1% per node. Also, as shown in [Okada et al., 1990], inter-node parameter...
fluctuations can lead to dispersion, causing decay as well as an introduction of additional non-soliton components. This leads to a change in the soliton parameter $\beta$, resulting in a decrease in soliton amplitude and velocity as they propagate through the lattice. Our circuit implementation used resistors and capacitors with 1% and 5% tolerances, respectively. Also, each of the double capacitor circuits used trim capacitances to match the transfer characteristics. A more detailed investigation of the effects of such perturbations on the induced soliton behavior might provide a better picture of the achieved accuracy of the circuit models. In the figure, there is also a small spike that appears in each of the diode currents near the time $t = -1$ ms. This results from the reset signal, $V_{sw}$, propagating down the lattice and resetting adjacent double capacitors at slightly different times.

### 3.5. Circuit model for discrete-KdV

Given the similarity between Eq. (13) for dKdV and Eq. (28) for the diode ladder circuit, we first consider a ladder of diodes with shunt capacitors for a dKdV circuit model. A similar analysis leads to the following set of equations

$$ \frac{dv_n(t)}{dt} = \frac{I_s}{C}(e^{(v_{n-1}(t) - v_n(t))/v_t} - e^{(v_n(t) - v_{n+1}(t))/v_t}), $$

which are similar in form to Eq. (13). However, there is no apparent means of decoupling the node voltage, $v_n(t)$ from $v_{n+1}(t)$ and $v_{n-1}(t)$ as would be required for dKdV. These node voltages can be decoupled in a sense through a realization of the discrete-KdV equation using two Toda circuits and the construction shown in Fig. 4. Although a dKdV circuit could be so constructed, the resulting circuitry would be twice as complex as the diode ladder circuit implementation of the Toda lattice. A much simpler implementation can be found by maintaining the aspects of the diode ladder that are useful, which is the exponential current relationship of the diodes, while removing the aspects which are troublesome, viz. the ladder interconnections.

Since the desired equations are of first-order, capacitor voltages can be used for state variables, i.e. $v_n(t)$ will be the voltage on the $n$th capacitor. Rather than assembling the capacitors in a ladder network, a collection of nodes with nearest-neighbor coupling as shown in Fig. 13 can be used. Each node maintains a voltage, $v_n(t)$, and also maintains a voltage that is proportional to $e^{v_n(t)}$, which can be accomplished with a voltage follower and a diode as shown in Fig. 13. Since the voltage follower mirrors the voltage on the capacitor, neglecting the voltage drop from the resistor, the voltage across the diode is approximately $v_n(t)$. Hence the current through the diode is $i_n(t) \approx I_s(e^{v_n(t)/v_t} - 1)$, where $v_t$ is the thermal voltage. All that remains is to construct a current source that is proportional to the difference in the exponential reference voltages of the neighboring nodes. If this current source is used to drive the capacitor as shown in Fig. 13, then the node voltage, $v_n(t)$, is governed by

$$ \dot{v}_n(t) = \frac{I_s}{C}(e^{v_{n-1}(t)/v_t} - e^{v_{n+1}(t)/v_t}). $$

The differential voltage controlled current source shown in Fig. 9 and used to drive the Toda ladder circuit can be used for the dKdV circuit as well. Therefore, the node capacitor voltages are governed by the discrete-KdV equation. The time scale of the circuit can be set by proper choice of the ratio $I_s/C$. Specifically, if $I_s = \alpha v_t C$, the node voltage satisfies

$$ \frac{d}{d\tau} \left( \frac{v_n(\tau)}{v_t} \right) = \left( e^{v_{n-1}(\tau)/v_t} - e^{v_{n+1}(\tau)/v_t} \right), $$

where $\tau = t/\alpha$. Thus $v_n(t)/v_t$ satisfies the discrete-KdV equation on a time-scale $t/\alpha$.

An HSPICE simulation of this circuit verifies the propagation of dKdV solitons. Since this circuit is of first-order, the state of the system is completely specified by the capacitor voltages. Rather than processing continuous-time signals as with the Toda lattice system, we can use this system to process discrete-time solitons as specified by $v_n$ at a fixed time, $t$. For the purposes of simulation, we consider the periodic dKdV equation by setting

---

**Fig. 13.** Collection of nodes for the discrete-KdV circuit.
Fig. 14. To the left, the normalized node capacitor voltages, $v_n(t)/v_t$ for each node is shown as a function of time. To the right, the state of the circuit is shown as a function of node index for five different sample times. The bottom trace in the figure corresponds to the initial condition.

$v_{n+1}(t) = v_0(t)$ and initializing the system with the discrete-time signal corresponding to a listing of node capacitor voltages. We can place a multisoliton solution in the circuit using inverse scattering techniques to construct the initial voltage profile. The single soliton solution to the dKdV system is given by

$$v_n(t) = \ln \left( \frac{\cosh(\gamma(n - 2) - \beta t) \cosh(\gamma(n + 1) - \beta t)}{\cosh(\gamma(n - 1) - \beta t) \cosh(\gamma n - \beta t)} \right),$$

(44)

where $\beta = \sinh(2\gamma)$. Shown in Fig. 14, is the result of an HSPICE simulation of the circuit with 30 nodes in a loop configuration, each with $2nF$ capacitors and diodes with a saturation current chosen to be $I_s = v_t \times 2$ nA. Thus, the time scale of the circuit is unity, $\alpha = 1$. The initial condition was set such that a soliton with $\gamma_1 = 2.5$ was placed on node 0 and a second soliton with $\gamma_2 = 2$ was placed on node 10. As with the diode ladder implementation of the Toda lattice equations, this circuit model is accurate over a wide range of soliton wavenumbers.

4. Noise Dynamics in Soliton Systems

In any eventual application of these circuits for generation and processing of soliton signals, it is important to understand the effects of random fluctuations on the dynamics of soliton systems. Such disturbances could take the form of additive noise or interference, circuit thermal noise, or modeling errors due to system deviation from the idealized soliton dynamics. In this section, we first assume that the diode ladder circuit is an accurate representation of the Toda lattice and investigate the effects of low-level additive noise at the input to the system. In Sec. 4.5 we discuss how the results are affected by the characteristics of the diode ladder circuit.

In the literature, several measures of the stability or robustness of such nonlinear systems and their soliton solutions have been investigated. In addition to early numerical work, studies like [Ooyama & Saito, 1970] have empirically investigated the stability of soliton solutions in the presence of additive corruption. Lindgren and Buratti [1969] have studied analytically the stability of solitons in the sine-Gordon equation through a linearization about a known soliton solution. Some success has also been achieved for such a study of one-dimensional nonlinear lattices by Flytzanis et al. [1993]. Many forms of perturbation theory and approximate linear analysis have also been applied to the nonlinear Schrödinger equation, demonstrating the viability of proposed telecommunication systems as well [Lai, 1993]. In [Benjamin, 1972] the Lyapunov stability of a single soliton is demonstrated in the fully nonlinear KdV equation.

There has also been increasing interest in the solvability of soliton equations in the presence of additive noise. This area of the literature concerns systems such as the “stochastic KdV equation” which is a rather restrictive setting in which noise is additive and is a function of time, while remaining a constant function of space,

$$u_t + 6uu_x + u_{xxx} = n(t).$$

(45)
This system can be shown to possess an exact soliton solution with a phase drift that is given by a Wiener process, when \( n(t) \) is a stationary white Gaussian process [Loginov, 1993; Wadati, 1983].

With the development of the inverse scattering framework and the discovery that many soliton systems were conservative Hamiltonian systems, many of the questions regarding the stability of soliton solutions are readily answered. For example, any solitons that are initially present in a system must remain present for all time, regardless of their interactions. Similarly, the dynamics of any nonsoliton components that are present in the system are uncoupled from the dynamics of the solitons. However, in the communication scenario discussed in [Singer et al.], soliton waveforms are generated and extracted from the circuit and then propagated over a noisy channel. During transmission, these waveforms are susceptible to additive corruption from the channel.

In this section, we will assume that soliton signals generated with the circuits described in Sec. 3 have been transmitted over an additive white Gaussian noise channel. We can then consider the effects of additive corruption on the processing of soliton signals with their nonlinear evolution equations. Two general approaches are taken to this problem. The approach taken in this paper primarily deals with linearized models and investigates the dynamic behavior of the noise component of signals containing an information bearing soliton signal and additive noise. The second approach, which is developed in [Singer et al.], is taken in the framework of inverse scattering and is based on some results from random matrix theory. Although the analysis techniques developed in this section are applicable to a large class of soliton systems, we focus our attention on the Toda lattice equations as a representative example.

### 4.1. Toda lattice small signal model

If a signal that is processed in a receiver circuit representing a Toda lattice contains only a small amplitude noise component, then the dynamics of the receiver can be approximated by a small signal model. Starting with the nonlinear transmission line model,

\[
\frac{d^2 V_n(t)}{dt^2} \ln(1+V_n(t)) = \frac{\omega_0^2}{4} (V_{n-1}(t) - 2V_n(t) + V_{n+1}(t)),
\]

and using the approximation, \( \ln(1+x) \approx x \), the lattice equations can be approximately described by the linear lattice equations

\[
\frac{d^2 V_n(t)}{dt^2} = \frac{\omega_0^2}{4} (V_{n-1}(t) - 2V_n(t) + V_{n+1}(t)),
\]

when the amplitude of \( V_n(t) \) is appropriately small. Since this model is linear, we may decompose solutions into harmonic components of the form

\[
V_n(t) = V_+ e^{j(kn - \omega t)} + V_- e^{j(kn + \omega t)}.
\]

From Eqs. (47) and (48) the frequency of a single forward propagating solution must satisfy the dispersion relation

\[
-\omega^2 = \frac{\omega_0^2}{4} (e^{-jk} - 2 + e^{jk}),
\]

which reduces to \( \omega = \omega_0 \sin(k/2) \). Therefore the lattice is dispersive, with frequency-dependent velocity,

\[
c(k) = \frac{\omega_0 \sin \left( \frac{k}{2} \right)}{k},
\]

or

\[
c(\omega) = \frac{\omega}{2 \sin^{-1} \left( \frac{\omega}{\omega_0} \right)}.
\]

Note that we can also write the dispersion relation as

\[
k = 2 \sin^{-1} \left( \frac{\omega}{\omega_0} \right),
\]

from which \( k \) is only real if \(|\omega| \leq \omega_0\), for which there are propagating waves. When \( \omega \) is outside this region, the wavenumber, \( k \), is complex, corresponding to evanescent waves of the form

\[
\omega = \omega_0 \cosh \left( \frac{\text{Im}(k)}{2} \right), \quad \text{Re}(k) = \pi.
\]

These solutions decay as they pass through the lattice,

\[
|V_n| = |V_+| e^{-2 \cosh^{-1}(\omega/\omega_0)n},
\]

which for \( \omega \gg \omega_0 \) corresponds to

\[
V_n = V_+ \left( \frac{-\omega_0^2}{4\omega^2} \right)^n.
\]

If we consider processing signals with an infinite linear lattice and obtain an input–output relationship, where a signal is input at the zeroth node and the output is taken as the voltage on the Nth node,
can write Eq. (46) as
\[
\frac{d^2}{dt^2} \ln(1+S_n+v_n) = \frac{\omega_0^2}{4} (S_n-1-2S_n+S_{n+1}+v_{n-1}-2v_n+v_{n+1}).
\] (59)

After factoring the argument of the logarithm and cancelling terms from both sides that correspond to the known soliton solution, we have an exact equation that is satisfied by the nonsoliton component,
\[
\frac{d^2}{dt^2} \ln \left(1 + \frac{v_n(t)}{1+S_n(t)}\right) = \frac{\omega_0^2}{4} (v_{n-1}(t)-2v_n(t)+v_{n+1}(t)),
\] (60)

which can be viewed as the fully nonlinear model with a time-varying parameter, \((1+S_n(t))\). As a result, over short time scales relative to \(S_n(t)\), we would expect this model to behave in a similar manner to the small signal model of Eq. (47). With \(v_n(t) \ll (1+S_n(t))\), we obtain
\[
\frac{d^2}{dt^2} \frac{v_n(t)}{1+S_n(t)} \approx \frac{\omega_0^2}{4} (v_{n-1}(t)-2v_n(t)+v_{n+1}(t)).
\] (61)

When the contribution from the soliton is small, Eq. (61) reduces to the linear system of Eq. (47). We would therefore expect that both before and after a soliton has passed through the lattice, the system essentially lowpass filters the noise. However, as the soliton is processed, there will be a time-varying component to the filter.

### 4.3. Simulation of the lattice in noise

To confirm the intuition developed through small signal analyses, we have simulated the fully nonlinear dynamics. We work with a finite-length lattice which is terminated with its linearized impedance as described in Sec. 3. We then focus on the dynamics of the small amplitude noise component in the response of the lattice to a signal containing a single soliton in white Gaussian noise with noise power \(N_0\).

Our primary interest in this section is to characterize the effects of additive noise in a receiver for a potential soliton modulation system. Since the bandwidth limitations of the receiver for any of the communication scenarios discussed in [Singer...
et al.], will restrict the possible range of soliton parameters, without loss of generality, we may assume that the receiver contains a lowpass filter followed by a Toda lattice circuit. We also assume that the bandwidth, $2\pi/\Delta$, of the lowpass filter is wide enough to pass the soliton component of the received signal completely. The input to the Toda lattice circuit, $V_0(t)$, then contains the soliton signal in lowpass Gaussian noise. Simulations were performed using a Runge–Kutta integration routine with a fixed step size, $\Delta$. To model the effects of the noise, an i.i.d. Gaussian random sequence, $w(k\Delta) \sim N(0, \sigma_w^2)$, was added to the samples of the input sequence $V_0(k\Delta)$ resulting in an effective white noise power of $N_0 = \Delta\sigma_w^2$.

Specifically, the circuit equations governing the resistance-terminated nonlinear LC ladder model are given in Eq. (46) for $n < N$ and $\omega_0 = 2/\sqrt{LC}$. At the termination, $n = N$, we have

$$\dot{V}_N(t) = \left(\frac{V_{N-1}(t) - V_N(t)}{LC} - \frac{\dot{V}_N(t)}{RC}\right)(1 + V_N(t))$$

$$+ \frac{\dot{V}_N^2(t)}{1 + V_N(t)}, \quad (62)$$

Writing Eq. (46) and (62) as $2N$ first-order differential equations, we obtain

$$\dot{V}_n(t) = W_n(t), \quad 1 \leq n \leq N \quad (63)$$

$$\dot{W}_n(t) = \frac{W_n^2(t)}{1 + V_n(t)} + \frac{1 + V_n(t)}{LC}$$

$$\times (V_{n+1}(t) - 2V_n(t) + V_{n-1}(t)), \quad \quad 1 \leq n < N$$

$$\dot{W}_N(t) = \left(\frac{V_{N-1}(t) - V_N(t)}{LC} - \frac{W_N(t)}{RC}\right)(1 + V_N(t))$$

$$+ \frac{W_N^2(t)}{1 + V_N(t)}.$$  

Note that since (62) and (63) are the exact nonlinear Toda lattice equations, their numerical integration will simulate both the nonlinear LC ladder and the diode ladder circuit implementations. However, while the diode ladder implementation with ideal components exactly matches the Toda lattice equations, the nonlinear LC ladder does not, even with ideal components. These simulations could have equally been carried out using diode ladder currents, with the substitution, $i_n(t)/I_s = V_n(t)/V_0$, and $\alpha I_s/v_t = 1/LC$.

From our linearized analyses, we anticipate that the response of the lattice to a soliton in small amplitude Gaussian noise will essentially contain the unperturbed soliton with an additional small amplitude lowpass Gaussian component. Through numerical simulation, in Fig. 16 we show the response of the fully nonlinear lattice to a single soliton at 20 dB signal-to-noise ratio, where the SNR is defined as

$$\text{SNR} = 10 \log \left(\frac{1}{N_0} \int_{-\infty}^{\infty} s(t)^2 dt\right), \quad (64)$$

where $s(t) = \beta^2 \text{sech}^2(\beta \tau)$ is a signal containing a single soliton waveform, and for the diode lattice, $\tau = t\sqrt{\alpha/v_t}$. As expected, the response to the lattice has the appearance of an unperturbed soliton with an additional lowpass perturbation.

As the SNR decreases, and correspondingly $v_n(t)/(1 + S_n(t))$ becomes significant relative to unity, the linear approximation no longer applies and the noise dynamics do not appear to be as simple as described by a lowpass filter. As shown in the simulation results in Fig. 17, the response to the noise term, $v_n(t)$, contains a contribution from nonsoliton components which is better handled in the framework of inverse scattering, as developed in [Singer et al.].
4.4. Noise correlation

The statistical correlation of the system response to the noise component can also be estimated from our linear analyses. From Sec. 4.1, the small signal model for the nonlinear lattice approximately satisfies the linear lattice equations, which have a magnitude-squared frequency response at the \( n \)th node, \( n \gg 1 \), of Eq. (57). Therefore, \( v_n(t) \) is zero mean and has an autocorrelation function given by

\[
R_{n,n}(\tau) = E\{v_n(t)v_n(t+\tau)\} \approx N_0 \frac{\sin(\omega_0\tau)}{\pi\tau}, \quad (65)
\]

and a variance \( \sigma_{v_n}^2 \approx N_0\omega_0/\pi \), for \( n \gg 1 \).

Although the autocorrelation of the noise at each node is only affected by the magnitude response of Eq. (56), the cross-correlation between nodes is also affected by the phase response. The cross-correlation between nodes \( m \) and \( n \) is given by

\[
R_{m,n}(\tau) = h_{n-m}(-\tau) \ast R_{m,m}(\tau), \quad (66)
\]

where \( h_m(\tau) \) is the inverse Fourier transform of \( H_m(j\omega) \) in Eq. (56). Since \( h_m(\tau) \ast h_m(-\tau) \) approaches the impulse response of an ideal lowpass filter for \( m \gg 1 \), we have

\[
R_{m,n}(\tau) \approx N_0 \frac{\sin(\omega_0\tau)}{\pi\tau} \ast h_{n-m}(\tau). \quad (67)
\]

In Fig. 18, \( R_{m,n}(\tau) \) is shown for \( n > m \gg 1 \). Note that for \( \omega \) small in Eq. (56), \( \sin^{-1}(\omega/\omega_0) \approx \omega/\omega_0 \), and the lattice looks like a pure delay of

\[
\alpha = 2(n - m)/\omega_0, \quad \text{corresponding to}
\]

\[
R_{m,n}(\tau) \approx \sin \left( \frac{\omega_0(\tau - \alpha)}{\pi(\tau - \alpha)} \right).
\]

This approximation is only valid in the low frequency limit and corresponds to the diagonal translation of the largest lobe of \( R_{m,n}(\tau) \) in Fig. 18.

For small amplitude noise, the correlation structure can be examined through the linear lattice, which acts as a dispersive lowpass filter. A corresponding analysis of the nonlinear system in the presence of solitons becomes prohibitive in closed form. However, we can explore the analyses numerically by linearizing the dynamics of the system about the known soliton trajectory.

To examine the correlation structure in the presence of soliton components, we use the state space framework of linear dynamic systems. The state space model comprises a linear system of finite dimension, \( N \), with state vector

\[
\mathbf{x}(t) = [x_0(t), \ldots, x_{N-1}(t)]^\top
\]

and dynamics in state space form,

\[
\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + b(t)u(t), \quad (68)
\]

where \( A(t) \) is an \( N \times N \) state transition matrix, \( b(t) \) is an \( N \times 1 \) vector, and \( u(t) \) is a scalar input. We consider \( u(t) \) to be a zero-mean, white Gaussian noise process with noise power \( \sigma_u^2 \).

For a linear system of the form (68), the state covariance matrix,

\[
P(t) = E\{(\mathbf{x}(t) - E\{\mathbf{x}(t)\})(\mathbf{x}(t) - E\{\mathbf{x}(t)\})^\top\}, \quad (69)
\]
Equations (72) can be written in the form,

\[ \dot{P}(t) = A(t)P(t) + P(t)A(t)^\top + b(t)v_n^2b(t)^\top. \]  

(70)

To limit the number of state variables in \( x \), we again terminate the nonlinear lattice with its linearized impedance. If we assume that the input to the nonlinear lattice is of the form,

\[ V_{\text{in}}(t) = V_0^0(t) + u(t), \quad (71) \]

where \( u(t) \) is a small amplitude white Gaussian noise process, and \( V_0^0(t) \) corresponds to a known soliton input, we may linearize the dynamics about the known response of the system.

By seeking a response of the nonlinear LC ladder model of the form \( V_n(t) = V_n^0(t) + v_n(t) \), and \( W_n(t) = W_n^0(t) + w_n(t) \), where \( V_n^0(t) \) and \( W_n^0(t) \) are the known responses to the input \( V_0^0(t) \), and \( v_n(t) \) and \( w_n(t) \) are the responses to the small amplitude noise component, \( u(t) \), we obtain

\[ \dot{V}_n^0 + \dot{v}_n = W_n^0 + w_n, \quad 1 \leq n \leq N, \]

\[ \dot{W}_n^0 + \dot{w}_n = \frac{(W_n^0 + w_n)^2}{1 + V_n^0 + v_n} + \frac{1 + V_n^0 + v_n}{LC} (V_{n+1} - 2V_n + V_{n-1} + v_{n+1} - 2v_n + v_{n-1}), \]

for \( 1 \leq n < N \), and

\[ \dot{W}_N^0 + \dot{w}_N = \left( \frac{V_{N-1} - V_N + v_{N-1} - v_N}{LC} - \frac{W_N + w_N}{RC} \right) (1 + W_N + w_N) \]

\[ + \frac{(W_N + w_N)^2}{1 + V_N + v_N}, \]

where \( V_0^0 = V_0^0(t) \) and \( v_0 = u(t) \). Cancelling terms that correspond to the known input and response and terms higher than first-order, we obtain

\[ \dot{v}_n = w_n^0, \quad 1 \leq n \leq N \]

(72)

\[ \dot{w}_n = \left[ \frac{V_{n+1} - 2V_n^0 + V_{n-1}}{LC} - \frac{(W_n^0)^2}{(1 + V_n^0)^2} - \frac{2(1 + V_n^0)}{LC} \right] v_n \]

\[ + \frac{1 + V_n^0}{LC} (v_{n+1} + v_{n-1}) + \frac{2W_n^0}{1 + V_n^0} w_n, \quad 1 \leq n < N \]

\[ \dot{w}_N = \left[ \frac{V_{N-1} - V_N^0}{LC} - \frac{W_N^0}{RC} \right] v_N + \left[ \frac{v_{N-1} - v_N}{LC} - \frac{w_N}{RC} \right] (1 + V_N^0) + \frac{2W_N^0}{1 + V_N^0} w_N. \]

Equations (72) can be written in the form,

\[ \begin{bmatrix} \dot{v}(t) \\ \dot{w}(t) \end{bmatrix} = A(t) \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} + b(t)u(t), \]

(73)

where \( v(t) = [v_1(t), \ldots, v_N(t)]^\top \), and \( w(t) = [w_1(t), \ldots, w_N(t)]^\top \).
From our earlier linearized analyses, the linear time-varying small signal model can be viewed over short time scales as linear and time-invariant with a slowly varying parameter. The resulting input–output transfer function can be viewed as a low-pass filter with time-varying cutoff frequency equal to \( \omega_0 \) when a soliton is far from the node, and to \( \omega_0 \sqrt{1 + V_n^2} \) as a soliton passes through. Thus, we would expect the variance of the node voltage to rise from a nominal value as a soliton passes through.

We numerically integrate the corresponding Riccati equation, Eq. (70), for the node covariance and in Fig. 19, the resulting variance of the noise component on each node is shown. In this example, the input to the lattice was a periodically repeated single soliton with an initial SNR of 30 dB. Since the lattice was assumed initially at rest, there is a startup transient, as well as an initial spatial transient at the beginning of the lattice, after which we see that the variance of the noise is amplified from the nominal variance as each soliton passes through, confirming our earlier intuition.

### 4.5. Noise dynamics for the diode ladder circuit

The prior analyses developed in this section apply to the Toda lattice when the noise component of the signal can be considered small in comparison to the remaining arguments of the logarithm in

\[
\frac{d^2}{dt^2} \ln \left( 1 + \frac{V_n(t)}{V_0} \right) = -\frac{1}{LC_0 V_0} \left( V_{n-1}(t) - 2V_n(t) + V_{n+1}(t) \right). \tag{74}
\]

When \( V_n(t) \) is small as compared with \( V_0 \), then Eq. (74) behaves like a linear LC ladder. However, for the diode ladder circuit which satisfies,

\[
\frac{d^2}{dt^2} \ln \left( 1 + \frac{I_n(t)}{I_s} \right) = \frac{\alpha}{v_t} (I_{n-1}(t) - 2I_n(t) + I_{n+1}(t)), \tag{75}
\]

for a similar small signal analysis, \( I_n(t) \) would have to be small in comparison to the saturation current, \( I_s \). This would either require diodes with an unusually large saturation current or very small signal levels.

For a solution containing a soliton signal, \( I_n(t) \), and a small amplitude noise signal, \( i_n(t) \), an exact expression for the small amplitude component is given by

\[
\frac{d^2}{dt^2} \ln \left( 1 + \frac{i_n(t)}{I_n(t) + I_s} \right) = \frac{\alpha}{v_t} (i_{n-1}(t) - 2i_n(t) + i_{n+1}(t)). \tag{76}
\]

The linearization that results from the assumption that the current \( i_n(t) \) is small in comparison to the saturation current is tantamount to replacing the diodes with their equivalent linearized resistance, \( R_{eq} = v_t/I_s \), which is on the order of \( 8 \) kΩ for the range of circuit parameters used in Sec. 3. Since the soliton pulse-widths considered were on the order of \( \mu s \) for mA amplitudes, the bandwidth of the small signal model is extremely narrow in comparison to that of the soliton.

Observations of our circuit implementation of the diode ladder circuit seem to indicate that this bandwidth is too narrow to explain the level of higher-frequency circuit noise present. This is partially explained by the change in the cutoff frequency as solitons are processed through a node. That is, over regions where the soliton component is significant, the equivalent resistance of the diode becomes \( R_{eq} = v_t/I_n(t) \), which is on the order of \( 25 \) Ω for a 1 mA soliton. The effect this has on the linearized lattice is to make the lattice effectively an all-pass filter in the vicinity of propagating solitons.

As a practical matter, we note that there appears to be a small amount of diode leakage current present in the circuit implementation and will explore the effect of a small bias current on the dynamics of both the soliton components and the
small amplitude perturbation. For a solution containing a soliton, \( I_n(t) \), a small amplitude component, \( i_n(t) \), and a small bias current, \( I_b \), the resulting system dynamics are

\[
\frac{d^2}{dt^2} \ln \left( 1 + \frac{I_n + i_n + I_b}{I_s} \right) = \frac{\alpha}{v_t} (I_{n-1} - 2I_n + I_{n+1} + i_{n-1} - 2i_n + i_{n+1}),
\]

which reduces to

\[
\frac{d^2}{dt^2} \ln \left( 1 + \frac{i_n(t) + I_b}{I_n(t) + I_s} \right) = \frac{\alpha}{v_t} (i_{n-1}(t) - 2i_n(t) + i_{n+1}(t)).
\]  

(77)

When the noise component \( i_n(t) \) is small as compared with \( I_b \), and away from the peaks of the soliton signal, \( I_n(t) < I_b \), the dynamics further reduce to

\[
\frac{d^2}{dt^2} i_n(t) \approx \frac{I_b \alpha}{v_t} (i_{n-1}(t) - 2i_n(t) + i_{n+1}(t)),
\]

(79)

where the diodes are replaced by their linearization about the bias current, \( R_{eq} = v_t/I_b \), leading to an increase in the bandwidth of the effective lowpass filter.

In summary, due of the scaling of the diode ladder circuit, in order for the linear analyses to hold, the noise must be small in comparison to the diode saturation current, \( I_s \). When a soliton is present, if the noise is small compared with the soliton, then a linear model can hold as with the LC ladder. When there is no soliton, if there is a small bias current that is larger than the noise, this can also lead to a simple linear model. When there is neither a bias nor a soliton present, if the noise is not small as compared with the diode saturation current, then the noise satisfies the fully nonlinear system. The resulting disturbance is better described in terms of inverse scattering and leads to the problem of determining the spectrum of random linear operators, or random matrices [Singer et al.]

5. Conclusions

In this paper, we have developed a framework for exploring the generation and processing of soliton signals using analog circuits. We have taken the viewpoint of using solitons as carrier signals for transmission over linear, rather than nonlinear channels. The nonlinear evolution equations can then be viewed as specialized processors of this class of signals, which are naturally suited to performing a number of complex signal processing tasks. For example, these systems can efficiently generate soliton signals and can perform the nonlinear signal separation of multisoliton carriers necessary for multiplexing and demultiplexing multiple users in the soliton communications context presented in [Singer et al.].

Focusing specifically on two soliton systems, the Toda lattice and the discrete-KdV equation, we develop new electrical analogs for the generation and processing of soliton signals. Although analog circuit models have been previously developed for a variety of nonlinear wave equations in general, and for the Toda lattice in particular, our diode ladder implementation is the first direct electrical analog of this soliton system. Further, this appears to be the first circuit model of the Toda lattice which is sufficiently accurate to demonstrate true over-taking soliton interactions over a small number of nodes. The diode ladder circuit was implemented in hardware using standard components and provides a platform for the further development and testing of real-time soliton processing techniques. We have also developed a new circuit model for the discrete-KdV equation; a nonlinear system which was largely ignored for having no prior electrical or mechanical analog. The discrete-KdV circuit provides a framework for processing discrete-time soliton signals.

In a companion paper [Singer et al.], we show the potential for wireless multisiton communication techniques which appear to simultaneously reduce the transmitted signal energy and enhance communication performance. To assess the efficacy and the robustness of these communication techniques in the presence of background noise, we have analyzed the effects of small amplitude disturbances on the processing of soliton signals in the Toda lattice and characterized the statistics of the noise as it is processed. We have shown that in a high SNR white Gaussian noise background, the dynamics of soliton signals are practically unperturbed. Also, the noise component of the received signal remains essentially lowpass and Gaussian.
References