On U-Statistics and Compressed Sensing I: Non-Asymptotic Average-Case Analysis

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Abstract

Hoeffding's U-statistics model combinatorial-type matrix parameters (appearing in CS theory) in a natural way. This paper proposes using these statistics for analyzing random compressed sensing matrices, in the non-asymptotic regime (relevant to practice). The aim is to address certain pessimisms of worst-case *restricted isometry* analyses, as observed by both Blanchard & Dossal, et. al.

We show how U-statistics can obtain average-case analyses, by relating to *statistical restricted isometry property* (StRIP) type recovery guarantees. However unlike standard StRIP, random signal models are not required; the analysis used here holds in the *almost sure* (probabilistic) sense. For Gaussian/bounded entry matrices, we show that both ℓ_1 -minimization and LASSO essentially require on the order of $k \cdot [\log((n-k)/u) + \sqrt{2(k/n) \log(n/k)}]$ measurements to respectively recover at least $1-5u$ fraction, and $1-4u$ fraction, of the signals. Noisy conditions are considered. Empirical evidence suggests our analysis to compare well to Donoho & Tanner's recent large deviation bounds for ℓ_0/ℓ_1 -equivalence, in the regime of block lengths $1000 \sim 3000$ with high undersampling ($50 \sim 150$ measurements); similar system sizes are found in recent CS implementation.

In this work, it is assumed throughout that matrix columns are independently sampled.

Index Terms

approximation, compressed sensing, U-statistics, random matrices

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I. INTRODUCTION

Compressed sensing (CS) analysis involves relatively recent results from random matrix theory [1], whereby recovery guarantees are framed in the context of matrix parameters known as *restricted isometry constants*. Other matrix parameters are also often studied in CS. Earlier work on sparse approximation considered a matrix parameter known as *mutual coherence* [2]–[4]. Fuchs’ work on *Karush-Kuhn-Tucker (KKT)* conditions for sparsity pattern recovery considered a parameter involving a matrix *pseudoinverse* [5], re-occurring in recent work [4], [6], [7]. Finally, the *null-space property* [8]–[10] is gaining recent popularity - being the parameter closest related to the fundamental compression limit dictated by *Gel’fand widths*. All above parameters share a similar feature, that is they are defined over subsets of a certain fixed size k . This combinatorial nature makes them difficult to evaluate, even for moderate block lengths n . Most CS work therefore involve some form of randomization to help the analysis.

While the celebrated $k \log(n/k)$ result was initially approached via asymptotics, *e.g.*, [1], [11], [12], implementations require finite block sizes. Hence, non-asymptotic analyses are more application relevant. In the same practical aspect, recent work deals with non-asymptotic analysis of *deterministic* CS matrices, see [4], [7], [13], [14]. On the other hand certain situations may not allow control over the sampling process, whereby the sampling may be inherently random, *e.g.*, prediction of clinical outcomes of various tumors based on gene expressions [6]. Random sampling has certain desirable simplicity/efficiency features - see [15] on data acquisition in the distributed sensor setting. Also recent hardware implementations point out energy/complexity-cost benefits of implementing *pseudo-random binary sequences* [16]–[18]; these sequences mimic statistical behavior. Non-asymptotic analysis is particularly valuable, when random samples are costly to acquire. For example, each clinical trial could be expensive to conduct an excessive number of times. In the systems setting, the application could be running on a tight energy budget - whereby processing/communication costs depend on the number of samples acquired.

This work is inspired by the *statistical* notion of the restricted isometry property (StRIP), initially developed for deterministic CS analysis [13], [14]. The idea is to relax the analysis, by allowing sampling matrix parameters (that guarantee signal recovery) to be satisfied for a *fraction* of subsets. Our interest is in average-case notions in the context of randomized sampling, reason being that certain pessimisms of worst-case restricted isometry analyses have been observed in past works [12], [19], [20]. On the other hand in [21], Donoho & Tanner remarked on potential benefits of the above average-case notion, recently pursued in an adaptation of a previous asymptotic result [22]. In the multichannel setting, average-case notions are employed to make analysis more tractable [23], [24]. In [25] a simple thresholding algorithm

is analyzed via an average coherence parameter. However the works in this respect are few, most random analyses are of the worst-case type, see [12], [20], [26], [27]. We investigate the unexplored, with the aim of providing new insights and obtaining new/improved results for the average-case.

Here we consider a random analysis tool that is well-suited to the CS context, yet seemingly left untouched in the literature. Our approach differs from that of deterministic matrices, where average-case analysis is typically made accessible via mutual coherence, see [13], [14], [17]. For random matrices, we propose an alternative approach via *U-statistics*, which do not require random signal models typically introduced in StRIP analysis, see [13], [24], [25]; here, the results are stated in the *almost sure* sense. U-statistics apply naturally to various kinds of non-asymptotic CS analyses, since they are designed for combinatorial-type parameters. Also, they have a natural average-case interpretation, which we apply to recent recovery guarantees that share the same average-case characteristic. Finally thanks to the wealth of U-statistical literature, the theory developed here is open to other extensions, *e.g.*, in related work [28] we demonstrate how U-statistics may also perform worst-case analysis.

Contributions: average-case analyses are developed based on U-statistics, which are i) empirically observed to have good potential for predicting CS recovery in non-asymptotic regimes, and ii) theoretically obtain measurement rates that incorporate a non-zero failure rate (similar to the $k \log(n/k)$ rate from worst-case analyses). We utilize a U-statistical concentration theorem, under the assumption that the matrix columns are independently sampled. The large deviation error bound holds *almost surely* (for sufficiently large n , see Theorem 1). No random signal model is needed, and the error is of the order $(n/k)^{-1} \log(n/k)$, whereby k is the U-statistic kernel size (and k also equals sparsity level). Gaussian/bounded entry matrices are considered. For concreteness, we connect with StRIP-type guarantees (from [6], [7]) to study the fraction of recoverable signals (*i.e.*, average-case recovery) of: i) ℓ_1 -minimization and ii) *least absolute shrinkage and selection operator (LASSO)*, under noisy conditions. For both these algorithms we show $\text{const} \cdot k [\log((n-k)/u) + \sqrt{2(k/n) \log(n/k)}]$ measurements are essentially required, to respectively recover at least $1 - 5u$ fraction (Theorem 2), and $1 - 4u$ fraction (Theorem 3), of possible signals. This is improved to $1 - 3u$ fraction for the noiseless case. Here $\text{const} = \max(4/(a_1 a_2)^2, 2c_1/(0.29 - a_1)^2)$ for to be specified constants a_1, a_2, c_1 , where c_1 depends on the distribution of matrix entries. Note that the term $\sqrt{2(k/n) \log(n/k)}$ is at most 1 and vanishes with small k/n . Empirical evidence suggests that our approach compares well with recent results from Donoho & Tanner [22] - improvement is suggested for system sizes found in implementations [16], with large undersampling (*i.e.*, $m = 50 \sim 100$ and $n = 1000 \sim 3000$). The analysis here does show some pessimism in the size of const above, whereby $\text{const} \geq 4$ (we conjecture possible improvement). For

Gaussian/Bernoulli matrices, we find $\text{const} \approx 1.8$ to be inherently smaller, *e.g.*, for $k = 4$ this predicts recovery of 1×10^{-6} fraction with 153 measurements - empirically $m = 150$.

Note: StRIP-type guarantees [6], [7] seem to work well, by simply *not* placing restrictive conditions on the maximum eigenvalues of the size- k submatrices. Our theory seems to apply fairly well for various considered system sizes k, m, n (*e.g.*, Figure 4), however in *noisy* situations, a (relatively small) factor of $\sqrt{k}$ losses is seen without making certain maximum eigenvalue assumptions. The ℓ_1 -recovery estimation error is now bounded by a $\sqrt{k}$ factor of its best k -term approximation error (both errors measured using the ℓ_1 -norm). For LASSO, the non-zero signal magnitudes must now be bounded below by a factor $\sqrt{2k \log n}$ (with respect to noise standard deviation), as opposed to $\sqrt{2 \log n}$ in [6]. These losses are due to estimation techniques employed here.

Organization: We begin with relevant background on CS in Section II. In Section III we present a general U-statistical theorem for large-deviation (average-case) behavior. In Section IV the U-statistical machinery is applied to StRIP-type average-case recovery. We conclude in Section V.

Notation: The set of real numbers is denoted $\mathbb{R}$. Deterministic quantities are denoted using $a, \mathbf{a}$, or $\mathbf{A}$, where bold fonts denote vectors (*i.e.*, $\mathbf{a}$) or matrices (*i.e.*, $\mathbf{A}$). Random quantities are denoted using upper-case *italics*, where A is a random variable (RV), and $\mathbf{A}$ a random vector/matrix. Let $\Pr\{A \leq a\}$ denote the probability that event $\{A \leq a\}$ occurs. Sets are denoted using braces, *e.g.*, $\{1, 2, \dots\}$. The notation $\mathbb{E}$ denotes expectation. The notation i, j, ℓ, ω is used for indexing. We let $\|\cdot\|_p$ denote the ℓ_p -norm for $p = 1$ and 2 .

II. PRELIMINARIES

A. Compressed Sensing (CS) Theory

A vector $\boldsymbol{\alpha}$ is said to be k -sparse, if at most k vector coefficients are non-zero (*i.e.*, its ℓ_0 -distance satisfies $\|\boldsymbol{\alpha}\|_0 \leq k$). Let n be a positive integer that denotes block length, and let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$ denote a length- n signal vector with signal coefficients α_i . The *best k -term approximation* $\bar{\boldsymbol{\alpha}}_k$ of $\boldsymbol{\alpha}$, is obtained by finding the k -sparse vector $\bar{\boldsymbol{\alpha}}_k$ that has minimal approximation error $\|\bar{\boldsymbol{\alpha}}_k - \boldsymbol{\alpha}\|_2$.

Let Φ denote an $m \times n$ CS sampling matrix, where $m < n$. The length- m **measurement vector** denoted $\mathbf{b} = [b_1, b_2, \dots, b_m]^T$ of some length- n signal $\boldsymbol{\alpha}$, is formed as $\mathbf{b} = \Phi \boldsymbol{\alpha}$. Recovering $\boldsymbol{\alpha}$ from $\mathbf{b}$ is challenging as Φ possesses a *non-trivial null-space*. We typically recover $\boldsymbol{\alpha}$ by solving the (convex) ℓ_1 -**minimization** problem

$$\min_{\tilde{\boldsymbol{\alpha}} \in \mathbb{R}^n} \|\tilde{\boldsymbol{\alpha}}\|_1 \quad \text{s. t.} \quad \|\mathbf{b} - \Phi \tilde{\boldsymbol{\alpha}}\|_2 \leq \epsilon. \quad (1)$$

The vector $\tilde{\mathbf{b}}$ is a *noisy* version of the original measurements $\mathbf{b}$, and here ϵ bounds the noise error, *i.e.*, $\epsilon \geq \|\tilde{\mathbf{b}} - \mathbf{b}\|_2$. Recovery conditions have been considered in many flavors [2], [3], [11], [21], [22], and mostly rely on studying parameters of the sampling matrix Φ .

For $k \leq n$, the k -th **restricted isometry constant** δ_k of an $m \times n$ matrix Φ , equals the smallest constant that satisfies

$$(1 - \delta_k)\|\alpha\|_2^2 \leq \|\Phi\alpha\|_2^2 \leq (1 + \delta_k)\|\alpha\|_2^2, \quad (2)$$

for any k -sparse α in $\mathbb{R}^n$. The following well-known recovery guarantee is stated w.r.t. δ_k in (2).

Theorem A, c.f., [29] *Let Φ be the sensing matrix. Let α denote the signal vector. Let $\mathbf{b}$ be the measurements, *i.e.*, $\mathbf{b} = \Phi\alpha$. Assume that the $(2k)$ -th restricted isometry constant δ_{2k} of Φ satisfies $\delta_{2k} < \sqrt{2} - 1$, and further assume that the noisy measurements $\tilde{\mathbf{b}}$ satisfy $\|\tilde{\mathbf{b}} - \mathbf{b}\|_2 \leq \epsilon$. Let $\bar{\alpha}_k$ denote the best- k approximation to α . Then the ℓ_1 -minimum solution α^* to (1) satisfies*

$$\|\alpha^* - \alpha\|_1 \leq c_1\|\alpha - \bar{\alpha}_k\|_1 + c_2\epsilon,$$

for small constants $c_1 = 4\sqrt{1 + \delta_{2k}}/(1 - \delta_{2k}(1 + \sqrt{2}))$ and $c_2 = 2(\delta_{2k}(1 - \sqrt{2}) - 1)/(\delta_{2k}(1 + \sqrt{2}) - 1)$.

Theorem A is very powerful, on condition that we know the constants δ_k . But because of their combinatoric nature, computing the restricted isometry constants δ_k is NP-Hard [12]. Let $\mathcal{S}$ denote a size- k subset of indices. Let $\Phi_{\mathcal{S}}$ denote the size $m \times k$ submatrix of Φ , indexed on (column indices) in $\mathcal{S}$. Let $\sigma_{\max}^2(\Phi_{\mathcal{S}})$ and $\sigma_{\min}^2(\Phi_{\mathcal{S}})$ respectively denote the minimum and maximum, *squared-singular values* of $\Phi_{\mathcal{S}}$. Then from (2) if the columns ϕ_i of Φ are properly normalized, *i.e.*, if $\|\phi_i\|_2 = 1$, we deduce that δ_k is the smallest constant in $\mathbb{R}$ that satisfies

$$\delta_k \geq \max(\sigma_{\max}^2(\Phi_{\mathcal{S}}) - 1, 1 - \sigma_{\min}^2(\Phi_{\mathcal{S}})), \quad (3)$$

for all $\binom{n}{k}$ size- k subsets $\mathcal{S}$. For large n , the number $\binom{n}{k}$ is huge. Fortunately δ_k need not be explicitly computed, if we can estimate it after incorporating *randomization* [1], [11].

Recovery guarantee Theorem A involves worst-case analysis. If the inequality (3) is violated for *any* one submatrix $\Phi_{\mathcal{S}}$, then the *whole* matrix Φ is deemed to have restricted isometry constant larger than δ_k . A common complaint of such worst-case analyses is pessimism, *e.g.*, in [19] it is found that for $n = 4000$ and $m = 1000$, the restricted isometry property is not even satisfied for sparsity $k = 5$. This motivates the average-case analysis investigated here, where the recovery guarantee is relaxed to hold for a large “fraction” of signals (useful in applications that do not demand all possible signals to be

completely recovered). We draw ideas from the statistical StRIP notion used in deterministic CS, which only require “most” of the submatrices Φ_S to satisfy some properties.

In statistics, a well-known notion of a U-statistic (introduced in the next subsection) is very similar to StRIP. We will show how U-statistics naturally lead to average-case analysis.

B. U-statistics and StRIP

A function $\zeta : \mathbb{R}^{m \times k} \rightarrow \mathbb{R}$ is said to be a **kernel**, if for any $\mathbf{A}, \mathbf{A}' \in \mathbb{R}^{m \times k}$, we have $\zeta(\mathbf{A}) = \zeta(\mathbf{A}')$ if matrix $\mathbf{A}'$ can be obtained from $\mathbf{A}$ by *column reordering*. Let $\mathbb{R}_{[0,1]}$ be the set of real numbers bounded below by 0 and above by 1, i.e., $\mathbb{R}_{[0,1]} = \{a \in \mathbb{R} : 0 \leq a \leq 1\}$. U-statistics are associated with functions $g : \mathbb{R}^{m \times k} \times \mathbb{R} \rightarrow \mathbb{R}_{[0,1]}$ known as **bounded kernels**. To obtain bounded kernels g from indicator functions, simply use some kernel ζ and set $g(\mathbf{A}, a) = \mathbb{1}\{\zeta(\mathbf{A}) \leq a\}$ or $g(\mathbf{A}, a) = \mathbb{1}\{\zeta(\mathbf{A}) > a\}$, e.g. $\mathbb{1}\{\sigma_{\max}^2(\mathbf{A}) \leq a\}$.

Definition 1 (Bounded Kernel U-Statistics). *Let $\mathbf{A}$ be a random matrix with n columns. Let Φ be sampled as $\Phi = \mathbf{A}$. Let $g : \mathbb{R}^{m \times k} \times \mathbb{R} \mapsto \mathbb{R}_{[0,1]}$ be a bounded kernel. For any $a \in \mathbb{R}$, the following quantity*

$$U_n(a) \triangleq \frac{1}{\binom{n}{k}} \sum_{\mathcal{S}} g(\Phi_{\mathcal{S}}, a) \quad (4)$$

is a U-statistic of the sampled realization $\Phi = \mathbf{A}$, corresponding to the kernel g . In (4), the matrix $\Phi_{\mathcal{S}}$ is the submatrix of Φ indexed on column indices in $\mathcal{S}$, and the sum takes place over all subsets $\mathcal{S}$ in $\{1, 2, \dots, n\}$. Note, $0 \leq U_n(a) \leq 1$.

For $k \leq n$ and positive u where $u \leq 1$, a matrix Φ has **u -StRIP constant** δ_k , if δ_k is the smallest constant s.t.

$$(1 - \delta_k) \|\alpha\|_2^2 \leq \|\Phi_{\mathcal{S}} \alpha\|_2^2 \leq (1 + \delta_k) \|\alpha\|_2^2, \quad (5)$$

for any $\alpha \in \mathbb{R}^k$ and fraction u of size- k subsets $\mathcal{S}$. Note (5) is similar to (2) except $\Phi_{\mathcal{S}}$ replaces Φ . This StRIP notion coincides with [7]. Consider $\zeta(\mathbf{A}) = \max(\sigma_{\max}^2(\mathbf{A}) - 1, 1 - \sigma_{\min}^2(\mathbf{A}))$ where ζ is a kernel. Obtain a bounded kernel g by setting $g(\mathbf{A}, a) = \mathbb{1}\{\zeta(\mathbf{A}) > a\}$. Construct a U-statistic $U_n(\delta)$ of Φ of the form $U_n(\delta) = \binom{n}{k}^{-1} \sum_{\mathcal{S}} \mathbb{1}\{\zeta(\Phi_{\mathcal{S}}) > \delta\}$. Then if this U-statistic satisfies $U_n(\delta) = 1 - u$, the u -StRIP constant δ_k of Φ is at most δ , i.e., $\delta_k \leq \delta$.

To exploit apparent similarities between U-statistics and StRIP, we turn to two average-case guarantees found in the StRIP literature. In the sequel, the conditions required by these two guarantees, will be analyzed in detail via U-statistics - for now let us recap these guarantees. First, an ℓ_1 -minimization

recovery guarantee recently given in [7], is a StRIP-adapted version of the worst-case guarantee Theorem A. For any non-square matrix $\mathbf{A}$, let $\mathbf{A}^\dagger$ denote the **Moore-Penrose pseudoinverse**.¹ A vector $\boldsymbol{\beta}$ with entries in $\{-1, 1\}$ is termed a **sign vector**. For $\boldsymbol{\alpha} \in \mathbb{R}^n$, we write $\boldsymbol{\alpha}_S$ for the length- k vector supported on S . Let S_c denote the complementary set of S , i.e., $S_c = \{1, 2, \dots, n\} \setminus S$. The average-case guarantees require us to check conditions on Φ for fractions of subsets S , or **sign-subset** pairs $(\boldsymbol{\beta}, S)$.

Theorem B, c.f., Lemma 3, [7] *Let Φ be an $m \times n$ sensing matrix where ϕ_i denotes its i -th column. Let S be a size- k subset, and let $\boldsymbol{\beta} \in \{-1, 1\}^k$. Assume that Φ satisfies*

- *invertibility: for at least a fraction $1 - u_1$ of subsets S , the condition $\sigma_{\min}(\Phi_S) > 0$ holds.*
- *small projections: for at least a fraction $1 - u_2$ of sign-subset pairs $(\boldsymbol{\beta}, S)$, the condition*

$$\left| (\Phi_S^\dagger \phi_i)^T \boldsymbol{\beta} \right| \leq a_2 \text{ for every } i \notin S$$

holds where we assume the constant $a_2 < 1$.

- *worst-case projections: for at least a fraction $1 - u_3$ of subsets S , the following condition holds*

$$\|\Phi_S^\dagger \phi_i\|_1 \leq a_3 \text{ for every } i \notin S.$$

Then for a fraction $1 - u_1 - u_2 - u_3$ of sign-subset pairs $(\boldsymbol{\beta}, S)$, the following error bounds are satisfied

$$\begin{aligned} \|\boldsymbol{\alpha}_S^* - \boldsymbol{\alpha}_S\|_1 &\leq \frac{2a_3}{1 - a_2} \|\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}_k\|_1, \\ \|\boldsymbol{\alpha}_{S_c}^* - \boldsymbol{\alpha}_{S_c}\|_1 &\leq \frac{2}{1 - a_2} \|\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}_k\|_1, \end{aligned}$$

where $\boldsymbol{\alpha}$ is a signal vector that satisfies $\text{sgn}(\boldsymbol{\alpha}_S) = \boldsymbol{\beta}$, and $\bar{\boldsymbol{\alpha}}_k$ is the best- k approximation of $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}_k$ is supported on S , and finally $\boldsymbol{\alpha}^$ is the solution to (1) where the noiseless measurements $\mathbf{b}$ satisfy $\mathbf{b} = \Phi \boldsymbol{\alpha}$.*

The second guarantee is a StRIP-type recovery guarantee for the *LASSO* estimate, based on [6] (also see [7]). Consider recovery from noisy measurements

$$\tilde{\mathbf{b}} = \Phi \boldsymbol{\alpha} + \mathbf{z},$$

here $\mathbf{z}$ is a length- m noise realization vector. We assume that the entries z_i of $\mathbf{z}$ are sampled from a zero-mean Gaussian distribution with variance c_Z^2 . The LASSO estimate considered in [6], is the optimal solution $\boldsymbol{\alpha}^*$ of the optimization problem

$$\min_{\tilde{\boldsymbol{\alpha}} \in \mathbb{R}^n} \frac{1}{2} \|\tilde{\mathbf{b}} - \Phi \tilde{\boldsymbol{\alpha}}\|_2 + 2c_Z \cdot \theta_n \|\tilde{\boldsymbol{\alpha}}\|_1. \quad (6)$$

¹If $\mathbf{A}$ has full column rank, then $\mathbf{A}^\dagger = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$,

The ℓ_1 -regularization parameter is chosen as a *product* of two terms c_Z and θ_n , where we specify $\theta_n = (1+a)\sqrt{2\log n}$ for some positive a . What differs from convention is that the regularization depends on the noise standard deviation c_Z . We assume $c_Z > 0$, otherwise there will be no ℓ_1 -regularization.

Theorem C, c.f., [6] *Let Φ be the $m \times n$ sensing matrix. Let $\mathcal{S}$ be a size- k subset, and let $\beta \in \{-1, 1\}^k$.*

- *invertability: for at least a fraction $1 - u_1$ of subsets $\mathcal{S}$, the condition $\sigma_{\min}(\Phi_{\mathcal{S}}) > a_1$ holds.*
- *small projections: for at least a fraction $1 - u_2$ of subsets $\mathcal{S}$, same as Theorem B.*
- *invertability projections: for at least a fraction $1 - u_3$ of sign-subset pairs $(\beta, \mathcal{S})$, the following condition holds*

$$\|(\Phi_{\mathcal{S}}^T \Phi_{\mathcal{S}})^{-1} \beta\|_{\infty} \leq a_3.$$

Let c_Z denote noise standard deviation. Assume Gaussian noise realization $\mathbf{z}$ in measurements $\tilde{\mathbf{b}}$, satisfy

- i) *$\|(\Phi_{\mathcal{S}}^T \Phi_{\mathcal{S}})^{-1} \Phi_{\mathcal{S}}^T \mathbf{z}\|_{\infty} \leq (c_Z \sqrt{2\log n})/a_1$, for the constant a_1 in the invertability condition.*
- ii) *$\|\Phi_{\mathcal{S}_c}^T (\mathbf{I} - \Phi_{\mathcal{S}} \Phi_{\mathcal{S}}^{\dagger}) \mathbf{z}\|_{\infty} \leq c_Z 2\sqrt{\log n}$, where $\mathcal{S}_c$ is the complementary set of $\mathcal{S}$.*

For some positive a , assume that constant a_2 in the small projections condition, satisfies

$$(\sqrt{2}(1+a))^{-1} + a_2 < 1. \quad (7)$$

Then for a fraction $1 - u_1 - u_2 - u_3$ of sign-subset pairs $(\beta, \mathcal{S})$, the LASSO estimate α^ from (6) with regularization $\theta_n = (1+a)\sqrt{2\log n}$ for the same a above, will successfully recover both signs and supports of α , if*

$$|\alpha_i| \geq [a_1^{-1} + 2a_3(1+a)] \cdot c_Z \sqrt{2\log n} \text{ for all } i \in \mathcal{S} \quad (8)$$

In [6] it is shown that the noise conditions i) and ii) are satisfied with large probability at least $1 - n^{-1}(2\pi \log n)^{-\frac{1}{2}}$. Theorem C is often referred to as a *sparsity pattern recovery* result, in the sense that it guarantees recovery of the sign-subset pairs $(\beta, \mathcal{S})$ belonging to a k -sparse signal α . Fuchs established some earlier important results, see [5], [30], [31].

In Theorems B and C, observe that the *invertability* condition can be easily checked using an U-statistic; simply set the bounded kernel g as $g(\mathbf{A}, a_1) = \mathbb{1}\{\sigma_{\min}(\mathbf{A}) \leq a_1\}$ for some positive a_1 and measure the fraction $U_n(a_1) = u_1$. Other conditions require slightly different kernels, to be addressed in upcoming Section IV. The next section introduces the main U-statistical concentration theorem used in our analyses.

Remark 1. *We emphasize again that while Theorem B accounts for best- k approximation errors, it does not account for noisy measurements; the opposite occurs for Theorem C. This is due to certain difficulties*

Fig. 1. Gaussian measure. Concentration of U-statistic $U_n(a)$ for squared singular value $\sigma_{\min}^2$ and $\sigma_{\max}^2$ kernels g , see (9) and (10). Shown for $m = 25, k = 2$ and two values of $n = 25$ and 100 .

in average-case case that prevent mimicking Candes' original arguments in [1]. However, the U-statistic technique is general, and may apply to future average-case analyses that potentially account for both noise sources.

III. LARGE DEVIATION THEOREM: AVERAGE-CASE BEHAVIOR

Consider two bounded kernels g defined for $\mathbf{A} \in \mathbb{R}^{m \times k}$, corresponding to maximum/minimum squared singular values

$$g(\mathbf{A}, a) = \mathbb{1} \{ \sigma_{\max}^2(\mathbf{A}) \leq a \}, \text{ and} \quad (9)$$

$$g(\mathbf{A}, a) = \mathbb{1} \{ \sigma_{\min}^2(\mathbf{A}) \leq a \}, \quad (10)$$

where recall $\mathbb{1} \{ \cdot \}$ refers to an indicator function. Note that restricted isometry conditions (2) and (5) depend on both $\sigma_{\min}^2$ and $\sigma_{\max}^2$ behaviors, although the conditions in the previous StRIP-recovery guarantees Theorem B are explicitly imposed only on $\sigma_{\min}^2$. See [12], [32] for the different behaviors and implications of these two extremal eigenvalues. In this section we consider two U-statistics, corresponding separately to (9) and (10).

Let $\mathbf{A}_i$ denote the i -th column of $\mathbf{A}$, and assume $\mathbf{A}_i$ to be IID. For an bounded kernel g , let $p(a)$ denote the expectation $\mathbb{E}g(\mathbf{A}_S, a)$, i.e., $p(a) = \mathbb{E}g(\mathbf{A}_S, a)$ for any size- k subset S . Since $p(a) = \mathbb{E}U_n(a)$, thus the U-statistic mean $\mathbb{E}U_n(a)$ does not depend on block length n .

Fig. 2. Means $p(a) = \mathbb{E}U_n(a)$ for predicting the concentration of $U_n(a)$. Shown for the Gaussian case, (a) $m = 50$ and (b) $m = 150$.

Theorem 1. Let $\mathbf{A}$ be an $m \times n$ random matrix, whereby the columns $\mathbf{A}_i$ are IID. Let g be a bounded kernel that maps $\mathbb{R}^{m \times k} \times \mathbb{R} \rightarrow \mathbb{R}_{[0,1]}$ and let $p(a) = \mathbb{E}g(\mathbf{A}_S, a) = \mathbb{E}U_n(a)$. Let $U_n(a)$ be a U -statistic of the sampled realization $\Phi = \mathbf{A}$ corresponding to the bounded kernel g . Then almost surely for sufficiently large but finite n , the deviation $|U_n(a) - p(a)| \leq \epsilon_n(a)$ is bounded by an error term $\epsilon_n(a)$ that satisfies

$$\epsilon_n^2(a) = 2p(a)(1 - p(a)) \cdot (n/k)^{-1} \log(n/k). \quad (11)$$

Theorem 1 is shown by piecing together (5.5) in [33] and Lemma 2.1 in [34]. The proof is given in Appendix A. Figure 1 empirically illustrates this concentration result for g in (9) and (10), corresponding to $p(a) = \mathbb{E}g(\mathbf{A}_S, a) = \Pr\{\sigma_{\max}^2(\mathbf{A}_S) \leq a\}$ and $p(a) = \Pr\{\sigma_{\min}^2(\mathbf{A}_S) \leq a\}$. Empirical simulation of restricted isometries is very difficult, thus we chose small values $k = 2$, $m = 25$ and block lengths $n = 25$ and $n = 100$. For $n = 25$ the deviation $|U_{25}(a) - p(a)|$ is very noticeable for all values of a and both $\sigma_{\max}^2$ and $\sigma_{\min}^2$. However for larger $n = 100$, the deviation $|U_{100}(a) - p(a)|$ clearly becomes much smaller. This is predicted by vanishing error $\epsilon_n(a)$ given in Theorem 1, which drops as the ratio n/k increases. In fact if k is kept constant then the error behaves as $\mathcal{O}(n^{-1} \log n)$.

Table I reproduces² a sample of (asymptotic) estimates for both $\sigma_{\max}^2$ and $\sigma_{\min}^2$ cases, taken from [20]. These estimates are derived for worst-case analysis, under assumption that every entry A_{ij} of $\mathbf{A}$ is

²We point out that Bah actually defined two separate restricted isometry constants, each corresponding to $\sigma_{\min}^2$ and $\sigma_{\max}^2$ in [20]. In this paper to coincide the presentation with our discussion on squared singular values, their results will be discussed in the domain of $\sigma_{\min}^2$ and $\sigma_{\max}^2$.

TABLE I
ASYMPTOTIC LOWER AND UPPER BOUNDS ON WORST-CASE EIGENVALUES, [20]

		Minimum: $\sigma_{\min}^2$			Maximum: $\sigma_{\max}^2$		
		m/n			m/n		
		0.1	0.3	0.5	0.1	0.3	0.5
k/m	0.1	0.095	0.118	0.130	3.952	3.610	3.459
	0.2	0.015	0.026	0.034	5.587	4.892	4.535
	0.3	0.003	0.006	0.010	6.939	5.806	5.361

IID and Gaussian distributed (*i.e.*, A_{ij} is Gaussian with variance $1/m$). Table I presents the estimates according³ to fixed ratios k/m and m/n . To compare, Figure 2 shows the expectations $p(a) = \mathbb{E}U_n(a)$. The values $p(a)$ are interpreted as fractions, and as n/k becomes large $p(a)$ is approached by $U_n(a)$ within a stipulated error ϵ_n . Figure 2 is empirically obtained, though note that in Gaussian case for $p(a)$ we also have exact expressions [32], [35], and the *Bartlett decomposition* [36], available. Again $p(a)$ is a marginal quantity (*i.e.* does not depend on n) and simulation is reasonably feasible. In the spirit of non-asymptotics, we consider relatively small k, m values as compared to other works [19], [20]; these adopted values are nevertheless “practical”, in the sense they come an implementation paper [16].

Differences are apparent from comparing average-case (Figure 2) and worst-case (Table I) behavior. Consider $k/m = 0.3$ where Table I shows for all undersampling ratios m/n , the worst-case estimate of $\sigma_{\min}^2$ is very small, approximately 0.01. But for fixed $m = 50$ and $m = 150$, Figures 2(a) and (b) show that for respectively $k = 0.3 \cdot (150) = 15$ and $k = 45$, a large fraction of subsets $\mathcal{S}$ seem to have $\sigma_{\min}^2(\Phi_{\mathcal{S}})$ lying above 0.1. From Table I, the estimates for $\sigma_{\min}^2$ gets worse (*i.e.*, gets smaller) as m/n decreases. But the error $\epsilon_n(a)$ in Theorem 1 vanishes with larger n/k . For the other $\sigma_{\max}^2$ case, we similarly observe that the values in Table I also appear more “pessimistic”.

We emphasize that Theorem 1 holds regardless of distribution. Figure 3 is the counterpart figure for Bernoulli and Uniform cases (*i.e.*, each entry A_{ij} is respectively drawn uniformly from $\{-1/\sqrt{m}, 1/\sqrt{m}\}$, or $\{a \in \mathbb{R} : |a| \leq \sqrt{3/m}\}$), shown for $m = 50$. Minute differences are seen when comparing with previous Figure 2. For $k = 3$, we observe the fraction $p(a)$ corresponding to $\sigma_{\max}^2$ to be roughly 0.95 in the latter case, whereas in the former we have roughly 0.9 in Figure 3(a), and 0.88 in Figure 3(b).

Remark 2. Exponential bounds on $\Pr\{\min_{\mathcal{S}} \sigma_{\min}^2(\mathbf{A}_{\mathcal{S}}) < 1 - \delta\}$ and $\Pr\{\max_{\mathcal{S}} \sigma_{\max}^2(\mathbf{A}_{\mathcal{S}}) > 1 + \delta\}$ for

³The analysis in [20] was performed for the large limit of k, m and n , where both k/m and m/n approach fixed constants.

Fig. 3. Means $p(a) = \mathbb{E}U_n(a)$ for $m = 50$ and the (a) Bernoulli and (b) Uniform cases.

$\max(\delta, \sqrt{k/m}) < \sqrt{2}-1$, see (3), employed in worst-case analyses, give the optimal $m = \mathcal{O}(k \log(n/k))$ rate, see [1], [27], [37]. However the implicit constants are inherently not too small (i.e., these constants cannot be improved).

These comparisons motivate average-case analysis. Marked out on Figures 2 and 3 are the ranges for which $\sigma_{\max}^2$ and $\sigma_{\min}^2$ must lie to apply Theorem A (worst-case analysis). In the cases shown above, the observations are somewhat disappointing - even for small k values, a substantial fraction of eigenvalues lie outside of the required range. Thankfully, there exist average-case guarantees, e.g., previous Theorems B and C, addressed in the next section.

IV. U-STATISTICS AND AVERAGE-CASE RECOVERY GUARANTEES

A. Counting argument using U-statistics

Previously we had explained how the *invertability* conditions required by Theorems B and C naturally relate to U-statistics. We now go on to discuss the other conditions, whereby the relationship may not be immediate. We begin with the *projections* conditions, in particular the *worst-case projections* condition. For given Φ , we need to *upper bound* the fraction of subsets $\mathcal{S}$, for which there exists *at least one* column ϕ_j where $j \notin \mathcal{S}$, such that $\|\Phi_{\mathcal{S}}^{\dagger} \phi_j\|_{\infty}$ exceeds some value a . To this end, let $\mathcal{R}$ denote a size- $(k+1)$ subset, and $\mathcal{R} \setminus \{j\}$ is the size- k subset excluding the index j . Consider the bounded kernel $g : \mathbb{R}^{m \times (k+1)} \times \mathbb{R} \mapsto \mathbb{R}_{[0,1]}$ set as

$$g(\mathbf{A}, a) = \frac{1}{k+1} \sum_{j=1}^{k+1} \mathbb{1} \left\{ \|\mathbf{A}_{\mathcal{R} \setminus \{j\}}^{\dagger} \mathbf{a}_j\|_{\infty} > a \right\}, \quad (12)$$

where here $\mathcal{R} = \{1, 2, \dots, k+1\}$, and $\mathbf{a}_j$ denotes the j -th column of $\mathbf{A}$. Consider the U-statistic with bounded kernel (12). We claim that

$$\begin{aligned} (n-k) \cdot U_n(a) &= \frac{n-k}{(k+1)\binom{n}{k+1}} \sum_{\mathcal{R}} \sum_{j \in \mathcal{R}} \mathbb{1} \left\{ \|\Phi_{\mathcal{R} \setminus \{j\}}^\dagger \phi_j\|_\infty > a \right\}, \\ &= \frac{1}{\binom{n}{k}} \sum_{\mathcal{S}} \sum_{j \notin \mathcal{S}} \mathbb{1} \left\{ \|\Phi_{\mathcal{S}}^\dagger \phi_j\|_\infty > a \right\}, \end{aligned}$$

where the summations over $\mathcal{R}$ and $\mathcal{S}$ are over all size- $(k+1)$ subsets, and all size- k subsets, respectively. The first equality follows from Definition 1 and (12). The second equality requires some manipulation. First the coefficient $\binom{n}{k}^{-1}$ follows from the binomial identity $\binom{n}{k+1} \cdot (k+1) = \binom{n}{k} \cdot (n-k)$. Next for some subset $\mathcal{S}$ and index j , write the indicator $\mathbb{1} \left\{ \|\Phi_{\mathcal{S}}^\dagger \phi_j\|_\infty > a \right\}$ as $\mathbb{1}_{\mathcal{S},j}$ for brevity's sake. By similar counting that proves the previous binomial identity, we argue $\sum_{\mathcal{R}} \sum_{j \in \mathcal{R}} \mathbb{1}_{\mathcal{R} \setminus \{j\},j} = \sum_{\mathcal{S}} \sum_{j \notin \mathcal{S}} \mathbb{1}_{\mathcal{S},j}$, which then proves the claim. Imagine a grid of “pigeon-holes”, indexed by pairs $(\mathcal{S}, j)$, where $j \notin \mathcal{S}$. For each size- $(k+1)$ subset $\mathcal{R}$, we assign $k+1$ indicators $\mathbb{1}_{\mathcal{R} \setminus \{j\},j}$ to $k+1$ pairs $(\mathcal{S}, j)$. No “pigeon-hole” gets assigned more than once. In fact we infer from the binomial identity, that every “pigeon-hole” is in fact assigned exactly once, and argument is complete.

Similarly for the *small projections* condition, we define a different bounded kernel $g : \mathbb{R}^{m \times (k+1)} \times \mathbb{R} \mapsto \mathbb{R}_{[0,1]}$ as

$$g(\mathbf{A}, a) = \frac{1}{2^k(k+1)} \sum_{\ell=1}^{2^k} \sum_{j=1}^{k+1} \mathbb{1} \left\{ \left| (\mathbf{A}_{\mathcal{R} \setminus \{j\}}^\dagger \mathbf{a}_j)^T \boldsymbol{\beta}_\ell \right| > a \right\}, \quad (13)$$

where $\mathcal{R} = \{1, 2, \dots, k+1\}$, and $\mathbf{a}_j$ denotes the j -th column of $\mathbf{A}$, and $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_{2^k}$ enumerate all 2^k unique sign-vectors in the set $\{-1, 1\}^k$. By similar arguments as before, we can show for the U-statistic $U_n(a)$ of Φ corresponding to the bounded kernel (13) satisfies

$$(n-k) \cdot U_n(a) = \frac{1}{2^k \binom{n}{k}} \sum_{\ell=1}^{2^k} \sum_{\mathcal{S}} \sum_{j \notin \mathcal{S}} \mathbb{1} \left\{ \left| (\Phi_{\mathcal{S}}^\dagger \phi_j)^T \boldsymbol{\beta}_\ell \right| > a \right\},$$

For indicators $\mathbb{1}_{\mathcal{S},j}$, note that $\sum_{j \notin \mathcal{S}} \mathbb{1}_{\mathcal{S},j} \geq 1$ if *at least one* indicator satisfying $\mathbb{1}_{\mathcal{S},j} = 1$, and we proved the following.

Proposition 1. *Let $U_n(a_3)$ be the U-statistic of Φ , corresponding to the bounded kernel $g(\mathbf{A}, a_3)$ in (12). Then the fraction of subsets $\mathcal{S}$ of size- k , for which the worst-case projections condition is violated for some $a_3 \in \mathbb{R}$, is at most $(n-k) \cdot U_n(a_3)$. Similarly if $U_n(a_2)$ corresponds to $g(\mathbf{A}, a_2)$ in (13), the fraction sign-subset pairs $(\boldsymbol{\beta}, \mathcal{S})$, for which the small projections condition is violated for some $a_2 \in \mathbb{R}$, is at most $(n-k) \cdot U_n(a_2)$.*

Referring back to Theorem B, we point out that the *small projections* condition is more stringent than the *worst-case projections* condition. We mean the following: in the former case, the value a_2 must be chosen such that $a_2 < 1$; in the latter case, the value a_3 is allowed to be larger than 1, its size only affects the constant $2a_3/(1 - a_2)$ appearing in the error estimate $\|\alpha_S^* - \alpha_S\|_1$. In fact if the signal α is k -sparse, then $\|\alpha - \bar{\alpha}_k\|_1 = 0$ and the size of a_3 is inconsequential, *i.e.*, the *worst-case projections* condition is not required in this special case. In this special case, it is best to set $a_2 = 1 - \epsilon$ for some arbitrarily small ϵ . Theorem B is in fact a stronger version of Fuchs' early work on ℓ_0/ℓ_1 -equivalence [5]. In the same respect, Donoho & Tanner also produced early seminal results from counting faces of random polytopes [21], [22].

Figure 4 shows empirical evidence, where the k, m, n values are inspired by practical system sizes taken from an implementation paper [16]. These experiments consider Φ sampled from Gaussian matrices $\mathbf{A}$, *exactly* k -sparse signals with non-zero α_i sampled from $\{-1, 1\}$, and uses ℓ_1 -minimization recovery (1). Figure 4(a) plots simulated (sparsity pattern recovery) results for 3 measurement sizes $m = 50, 100$ and 150 and block sizes $n \geq 200$ and $n \leq 3000$. For example the contour marked “0.1”, delineates the k, n values for which recovery fails for a 0.1 fraction of (random) sparsity patterns (sign-subset pairs $(\beta, \mathcal{S})$). We examine the U-statistic $U_n(a_2)$ with kernel (13), related to the small projections condition. Since $\mathbf{A}$ has Gaussian distribution, we set $a_2 = 1$ in the kernel $g(\mathbf{A}, a_2)$, as $\Pr\{(\mathbf{A}_S^\dagger \mathbf{A}_i)^T \beta = 1\} = 0$ for any $(\beta, \mathcal{S})$ and $j \notin \mathcal{S}$. Figure 4(b) plots the expectation $(n - k) \cdot p(1)$, where $p(1) = \mathbb{E}U_n(1) = \mathbb{E}g(\mathbf{A}_{\mathcal{R}}, 1)$ for any size- $(k + 1)$ subset $\mathcal{R}$. Again the contour marked “0.1”, delineates the k, n values for which $(n - k) \cdot p(1) = 0.1$. Here the values $p(1)$ are empirical. We observe that both Figures 4(a) and (b) are remarkably close for fractions 0.5 and smaller. Figures 4(c) incorporates the large deviation error ϵ_n given in Theorem 1 (in doing so, we assume n sufficiently large). The bound is still reasonably tight for fractions ≤ 0.5 . Comparing with recent Donoho & Tanners' (also average-case) results for ℓ_1 -recovery (for only the noiseless case), taken from [22]. For fractions 0.5 and 0.01, we observe that for system parameters $m = 50$ and $n \leq 1000$ (chosen in hardware implementation [16]), we do not obtain reasonable predictions. For $m = 100$, the bounds [22] work only for very small block lengths $n \leq 300$. The only reasonable case here is $m = 150$, where the bounds [22] perform better than ours only for lengths $n \leq 400$ (*i.e.*, Figure 4(c) shows that for $n = 300$, the large deviation bounds predict a 0.01 fraction of size $k = 5$ unrecoverable sparsity patterns, but [22] predict a 0.01 fraction of size $k = 11$ unrecoverable sparsity patterns).

The above experiments suggest the deviation error $\epsilon_n(a)$ in Theorem 1 to be over-conservative. Fortunately in the next two subsections (pertaining to U-statistics treatise of ℓ_1 -recovery Theorem

Fig. 4. Gaussian case. Comparing (a) empirical results for ℓ_1 -minimization recovery, (b) mean parameter $(n - k) \cdot p(1)$ (empirically obtained), and (c) after accounting for large deviations (Thm. 1). We show cases $m = 50, 100$ and 150 . We also compare with Donoho & Tanners' (DT) predictions [22].

B (Section IV-B), and LASSO recovery Theorem C (Subsection IV-C)), this conservative-ness does not show up from a rate standpoint (it only shows up in implicit constants). In fact by empirically “adjusting” these constants, we find good measurement rate predictions (akin to moving from Figure 4(c) to (b)).

B. Rate analysis for ℓ_1 -recovery (Theorem B)

In worst-case analysis, it is well-known that it is sufficient to have measurements m on the order of $k \log(n/k)$, in order to have the restricted isometry constants δ_k defined by (2), satisfy the conditions in Theorem A. We now go on to show that for average-case, a similar expression for this rate can be obtained. To this end we require tail bounds on salient quantities. Such bounds have been obtained for the *small projections* condition, see [6], [7], [24], where typically an equiprobable distribution is assumed over the sign-vectors β_ℓ . To our knowledge these techniques were born from considering deterministic matrices. Since Φ is randomly sampled here, we proceed slightly differently (though essentially using similar ideas) without requiring this random signal model. For simplicity, the bound assumes zero mean matrix entries, either i) Gaussian or ii) bounded.

Proposition 2. *Let $\mathbf{A}$ be an $m \times n$ random matrix, whereby its columns $\mathbf{A}_i$ are identically distributed. Assume every entry A_{ij} of $\mathbf{A}$ has zero mean, i.e., $\mathbb{E}A_{ij} = 0$. Let every A_{ij} be either i) Gaussian with variance $1/m$, or ii) bounded RVs satisfying $|A_{ij}| \leq 1/\sqrt{m}$. Let the rows $[A_{i1}, A_{i2}, \dots, A_{in}]$ of $\mathbf{A}$ be*

IID.

Let $\mathcal{S}$ be a size- k subset, and let index ω be outside of $\mathcal{S}$, i.e., $\omega \notin \mathcal{S}$. Then for any sign vector β in $\{-1, 1\}^k$, we have

$$\Pr \left\{ \left| (\mathbf{A}_{\mathcal{S}}^{\dagger} \mathbf{A}_{\omega})^T \beta \right| > a \right\} \leq 2 \exp \left(-\frac{ma^2\delta}{2k} \right) + \Pr \{ \sigma_{\min}^2(\mathbf{A}_{\mathcal{S}}) \leq \delta \} \quad (14)$$

for any positive $\delta \in \mathbb{R}$.

Proposition 2 is used as follows (see Appendix B for proof). Recall that Proposition 1 allows us to upper bound the fraction u_2 of sign-subset pairs $(\beta, \mathcal{S})$ failing the *small projections* condition, with the (scaled) U-statistic $(n-k) \cdot U_n(a_2)$ with kernel g in (13) and $|\mathcal{S}| = k$. By Theorem 1 the quantity $(n-k) \cdot U_n(a_2)$ concentrates around $(n-k) \cdot p(a_2)$, where $p(a_2) = \mathbb{E}g(\mathbf{A}_{\mathcal{R}}, a_2)$, where g in (13) is defined for size- $(k+1)$ subsets $\mathcal{R}$. We use Proposition 2 to upper estimate $p(a_2)$ using the RHS of (14). Indeed verify that $p(a_2) = 2^{-k} \sum_{\ell} \Pr \{ |(\mathbf{A}_{\mathcal{S}}^{\dagger} \mathbf{A}_{\omega})^T \beta_{\ell}| > a_2 \}$ for any $\mathcal{S}$ and $\omega \notin \mathcal{S}$, and the bound (14) holds for any $\beta = \beta_{\ell}$. Now $p(a_2)$ is bounded by two terms. By $u_2 \leq (n-k) \cdot U_n(a_2)$, thus to have u_2 small, we should have the (scaled) first term $2(n-k) \cdot \exp(-ma_2^2\delta/(2k))$ of (14) to be at most some small fraction u . This requires

$$m \geq \text{const} \cdot k \log \left(\frac{n-k}{u} \right) \quad (15)$$

with $\text{const} = 2/(a_2^2\delta)$ (where we dropped a smaller $\log 2$ term that should appear on the RHS). Next, for $m \geq 2k$ and $\delta < (0.29)^2$, we can bound⁴ the second term $\Pr \{ \sigma_{\min}^2(\mathbf{A}_{\mathcal{S}}) \leq \delta \}$ of (14) by $\exp(-m \cdot (0.29 - \sqrt{\delta})^2/c_1)$ where c_1 is some constant, see [26], Theorem 5.39. Roughly speaking, $\sigma_{\min}^2(\mathbf{A}_{\mathcal{S}}) \geq 0.29$ with “high probability”. Figures 2 and 3 (in the previous Section III) empirically support this fact. Again to have u_2 small the second term of (14) must be small. This requires $(n-k) \cdot \exp(-m \cdot (0.29 - \sqrt{\delta})^2/c_1) \leq u$ for some small fraction u , in which it suffices to have m satisfy (15) with $\text{const} = c_1/(0.29 - \sqrt{\delta})^2$.

For the *invertability* condition in Theorem B, we also need to upper bound the corresponding fraction u_1 of size- k subsets $\mathcal{S}$. We simply use an U-statistic $U_n(a_1)$ with kernel $g(\mathbf{A}, a_1) = \mathbb{1} \{ \sigma_{\min}(\mathbf{A}) > a_1 \}$ for some positive a_1 (see also Theorem C). Here Proposition 1 is not needed. To make $p(a_1)$ small, where $p(a_1) = \mathbb{E}g(\mathbf{A}_{\mathcal{S}}, a_1)$, use the previous bound $p(a_1) \leq \exp(-m \cdot (0.29 - a_1)^2/c_1)$, where we

⁴For $m \geq 2k$, we have $\Pr \{ \sigma_{\min}(\mathbf{A}) < 1 - c \cdot 0.29 - t \} \leq \Pr \{ \sigma_{\min}(\mathbf{A}) < 1 - c \cdot \sqrt{k/m} - t \} \leq \exp(-mt^2/c_1)$ for some constants c, c_1 , where $\mathbf{A}$ has size $m \times k$ and with proper column normalization. For simplicity we drop the constant c in this paper; one simply needs to add c in appropriate places in the exposition. In particular for the Gaussian and Bernoulli cases $c = 1$, and $c_1 = 2$ and $c_1 = 16$, respectively, see Theorem B, [28].

set $a_1 = \sqrt{\delta}$ with $a_1 \leq 0.29$. Clearly $p(a_1)$ cannot exceed some fraction u , if m satisfies (15) with $\text{const} = c_1/(0.29 - a_1)^2$.

For the time being consider *exactly* k -sparse signals $\mathbf{a}$. In this special case the *worst-case projections* condition in Theorem B is superfluous (*i.e.*, with no consequence a_3 can be arbitrarily big) - only *invertability* and *small projections* conditions are needed. While we have yet to consider the large deviation error $\epsilon_n(a)$ from Theorem 1, doing so will not drastically change the rate. For $U_n(a)$ with kernel g and $p(a)$, where $p(a) = \mathbb{E}g(\mathbf{A}, a)$, almost surely

$$U_n(a) \leq p(a) + \epsilon_n(a) \leq (p(a))^{\frac{1}{2}} + \sqrt{2p(a)\omega^{-1}\log\omega} \leq (p(a))^{\frac{1}{2}} \left(1 + \sqrt{2\omega^{-1}\log\omega}\right) \quad (16)$$

where the second inequality follows because $p(a) \leq 1$, and by setting $\omega = n/k$. Taking log of the RHS, we obtain $(1/2)\log p(a) + \log(1 + \sqrt{2\omega^{-1}\log\omega})$. Note $\log(1 + \sqrt{2\omega^{-1}\log\omega}) \leq \sqrt{2\omega^{-1}\log\omega}$, since $\log(1 + \alpha) \leq \alpha$ holds for all positive α . For the *small projections* condition, bound $(p(a))^{\frac{1}{2}}$ by the sum of the square-roots of each term in (14). Then to have $u_2 \leq (n - k) \cdot U_n(a_2) \leq 2u$, it follows similarly as before, it suffices that

$$m \geq \text{const} \cdot k \left[\log \left(\frac{n - k}{u} \right) + \sqrt{2 \cdot (k/n) \log(n/k)} \right] \quad (17)$$

with $\text{const} = \max(4/(a_2^2\delta), 2c_1/(0.29 - \sqrt{\delta})^2)$ where we had set $\sqrt{\delta} = a_1$. For *invertability* condition do the same. To have $u_1 = U_n(a_1) \leq u$ it suffices that m satisfies (17) with the same const . Observe that the term $\sqrt{2 \cdot (k/n) \log(n/k)}$ is at most 1, and vanishes with high undersampling (small k/n). Hence (15) and (17) are similar from a rate standpoint.

We conclude the following: for exactly k -sparse signals the rate (17) suffices to recover at least $1 - 3u$ fraction of sign-subset $(\beta, \mathcal{S})$ pairs. While const in (17) must be at least 4 (recall that Figure 4(c) was somewhat pessimistic), for matrices with Gaussian entries we empirically find that const is inherently smaller, whereby $\text{const} \approx 1.8$. This is illustrated in Figure 5, for two fractions 0.1 and 0.01 of unrecoverable sign-subset pairs. We observe good match with simulation results shown in the previous Figure 4(a), and quantities⁵ $(n - k) \cdot p(1)$ plotted in Figure 4(b). For example, $m = 150$ suffices for a 0.01 fractional recovery failure, for $n = 300 \sim 1000$ and $k = 6 \sim 7$, and for 0.1 fraction then $k = 7 \sim 10$. We conjecture possible improvement for const .

⁵Comparing (17) and (15) and the respective expressions for const , dropping const from 4 to 1.8 is akin to ignoring the deviation error $\epsilon_n(a)$. This, and as Figure 4 suggests, the U-statistic “means” $(n - k) \cdot p(1)$ seem to predict recovery remarkably well, with similar rates to (17), and inherent const smaller than that derived here.

Fig. 5. Measurement rates predicted by equation (17), with const taken to equal 1.8, required to recover at least $1 - 3u = 0.9$ and 0.99 fractions of sign-subset pairs (β, S) (when the signal is k -sparse), shown respectively in (a) and (b).

In the more general setting for *approximately* k -sparse signals, we can also have rate (17). To see this, observe that Proposition 2 also delivers an exponential bound for the *worst-case projections* condition, see (12). This is because $\|\mathbf{A}_S^\dagger \mathbf{A}_\omega\|_1 = \max_{\ell: 1 \leq \ell \leq 2^k} |(\mathbf{A}_S^\dagger \mathbf{A}_\omega)^T \beta_\ell|$, and we take a union bound over 2^k terms. Set $a_3 = a_2 \sqrt{k}$, where a_2 and a_3 respectively correspond to *small projections* and *invertability* conditions. Then we proceed similarly as before to show⁶ that the rate for recovering at least $1 - 5u$ fraction of (β, S) pairs suffices to be (17). The following is the main result summarizing the exposition so far.

Theorem 2. Let Φ be an $m \times n$ matrix, where assume n sufficiently large for Theorem 1 to hold. Sample $\Phi = \mathbf{A}$ whereby the entries A_{ij} are IID, and are Gaussian or bounded (as stated in Proposition 2). Then all three conditions in ℓ_1 -recovery guarantee Theorem B for (β, S) with $|S| = k$, with the invertability condition taken as $\sigma_{\min}(\Phi_S) \geq a_1$ with $a_1 \leq 0.29$. and with $a_3 = a_1 \sqrt{k}$, are satisfied for $u_1 + u_2 + u_3 = 5u$ for some small fraction u , if m is on the order of (17) with $\text{const} = \max(4/(a_1 a_2)^2, 2c_1/(0.29 - a_1)^2)$, and c_1 depends on the distribution of A_{ij} 's. Note $\text{const} \geq 4$.

In the exactly k -sparse case where only the first 2 conditions are required, this improves to $u_1 + u_2 = 3u$.

We end this subsection with two comments on the rate (17) derived here for average-case analysis.

⁶We used an assumption that $(n - k)/u$ is suitably larger than 2.

Fig. 6. Empirical LASSO recovery performance, Bernoulli case. In (a) the non-zero signal magnitudes $|\alpha_i|$ equal 1, and in (b) they are in $\mathbb{R}_{[0,1]}$. Noise variances denoted c_Z^2 .

Firstly (17) is very similar to that of $k \log(n/k)$ for worst-case analysis. This justifies the counting employed in previous Subsection IV-A, Proposition 1, and is reassuring since we know that worst-case analysis provides the optimal rate [1], [11]. Secondly to have (17) hold for the approximately k -sparse case, we lose a factor of $\sqrt{k}$ in the error estimate $\|\alpha_S^* - \alpha_S\|_1$, as compared to worst-case Theorem A. This is because we need to set $a_3 = a_2\sqrt{k}$, as mentioned in the previous paragraph. However, the average-case analysis here achieves our primary goal, that is to predict well for system sizes k, m, n when worst-case analysis becomes too pessimistic.

C. Rate analysis for LASSO (Theorem C)

Next we move on to the LASSO estimate of [6]. Recall from (6) that the regularizer depends on the noise standard deviation c_Z , and the term $\theta_n = (1 + a)\sqrt{2\log n}$ that depends on block length n and some non-negative constant a that we set. This constant a impacts performance [6]. For matrices with Bernoulli entries, Figure 6 shows recovery failure rates for two data sets $m = 50, n = 1000$ and $m = 150, n = 1000$; the sparsity patterns (sign-subset pairs $(\beta, \mathcal{S})$) were chosen at random, and failure rates are shown for various sparsity values k , and noises c_Z . In Figure 6(a) we set $a = 0$, and in (b) we set $a = 1$. Also, in (a) the non-zero signal magnitudes $|\alpha_i|$ are in $\{1, -1\}$, and in (b) they are in $\mathbb{R}_{[0,1]}$. The performances are clearly different. “Threshold-like” behavior is seen in (a) for both data sets, whereby the performances stay the same for c_Z in the range $5 \times 10^{-2} \sim 1 \times 10^{-4}$, and then catastrophically failing for $c_Z = 1 \times 10^{-1}$. However in (b), for various c_Z the performances seem to be limited by a “noise-floor”. We see that in the noiseless limit (more specifically when $c_Z \rightarrow 0$), the performances become the same. In this subsection, we apply U-statistics on the various conditions of Theorem C, in particular the *invertability* and *small projections* conditions have already been discussed in the previous subsection. We account for the observations in Figure 6.

In the noiseless limit, the previously derived rate (17) holds. Here, the regularizer in (6) becomes so small that a (equivalently θ_n) does not matter. As mentioned in [5], LASSO then becomes equivalent to ℓ_1 -minimization (1), hence the (noiseless) performances in Figures 6(a) and (b) are the same. That is, in this special case the rate (17) suffices to recover at least $1 - 3u$ fraction of $(\beta, \mathcal{S})$. To test, take $k = 4$, $n = 3000$, and fraction $1 - 3u = 1 - 6 \times 10^{-6}$, and with $\text{const} = 1.8$ gives 153, close to m here which is set to 150.

In the noisy case, we are additionally concerned with the noise conditions i) and ii), conditions (7) and (8), and *invertability projections*. Recall that the noise conditions are satisfied with probability $1 - n^{-1}(2\pi \log n)^{-\frac{1}{2}}$, that goes to 1 superlinearly [6]. The remaining conditions are influenced by the value a set in the θ_n regularization term in (6).

In condition (7), the value a sets the maximal value for a_2 (when $a = 0$ then $a_2 < 0.2929$, and when $a = 1$ then $a_2 < 0.6464$). This affects the *small projections* condition, to which constant a_2 belongs, which in turn affects performance. However from a rate standpoint (17) still holds, only now the value of const (which has the term $4/(a_2^2\delta)$) becomes larger.

In condition (8), the value a affects the size of the term $a_1^{-1} + 2a_3(1 + a)$. The larger a is, the more often (8) fails to satisfy. Here there are two constants a_1 and a_3 . Recall a_1 belongs to the *invertability*

condition discussed in the previous subsection, which holds with rate (17) with $\text{const} = 2c_1/(0.29 - a_1)^2$ and $a_1 \leq 0.29$. Consider the case where the non-zero signal magnitudes $|\alpha_i|$ are independently drawn from $\mathbb{R}_{[0,1]}$. Then we observe $(\min_{i \in \mathcal{S}} |\alpha_i|) < t$ with probability $1 - (1 - t)^k$ where $t \in \mathbb{R}_{[0,1]}$ and $|\mathcal{S}| = k$. For t set equal to the RHS of (8), this gives the probability that condition (8) fails. Figure 6(b) shows good empirical match when setting $a_1 = 0.29$ and $a_3 = 1$, where the dotted curves predict the “error-floors” for various k , measurements $m = 50$ and $m = 150$, and noise c_Z . In the other case where $|\alpha_i| = 1$ (as in Figure 6(a)), condition (8) remains un-violated as long as c_Z (and a_1, a_3, n) allow the RHS to be smaller than 1. Figure 6(a) suggests that for the appropriate choices for a_1, a_3 , condition (8) is always un-violated when $c_Z \leq 5 \times 10^{-2}$, and violated when $c_Z \geq 1 \times 10^{-1}$.

The constant a_3 belongs to the remaining *invertability projections* condition. The fraction u_3 of size- k subsets *failing* the *invertability projections* condition for some a_3 , can be addressed using U-statistics. Consider the bounded kernel $g : \mathbb{R}^{m \times k} \times \mathbb{R} \rightarrow \mathbb{R}_{[0,1]}$, set as

$$g(\mathbf{A}, a) = \frac{1}{2^k} \sum_{\ell=1}^{2^k} \mathbb{1} \left\{ (\mathbf{A}^T \mathbf{A})^\dagger \boldsymbol{\beta}_\ell > a \right\} \quad (18)$$

where $\boldsymbol{\beta}_\ell \in \{-1, 1\}^k$ and $(\mathbf{A}^T \mathbf{A})^\dagger$ is the pseudoinverse of $\mathbf{A}^T \mathbf{A}$. Then $u_3 = U_n(a_3)$, and as before Theorem 1 guarantees the upper bound (16), which depends on $p(a_3)$ where $p(a_3) = \mathbb{E}g(\mathbf{A}_\mathcal{S}, a_3)$.

We go on to discuss a bound on $p(a_3)$ under some general conditions. In [6], analysis on $p(a_3)$ (see Lemma 3.5) requires $\sigma_{\max}^2(\mathbf{A}_\mathcal{S}) \leq 1.5$, a condition not explicitly required in Theorem C. Also, empirical evidence suggests not to assume that $\sigma_{\max}^2(\mathbf{A}_\mathcal{S}) \leq 1.5$. For $m = 150$ and $k = 5$ we see from Figure 6 that (in the noiseless limit) the *failure* rate is on the order of 1×10^{-4} , but in Figure 2(b) we see $\sigma_{\max}^2(\mathbf{A}_\mathcal{S}) > 1.5$ occurs with much larger fraction 0.1. Hence we take a different approach. Using ideas behind Bauer’s generalization of *Wielandt’s inequality* [38], the following proposition allows $\sigma_{\max}^2(\mathbf{A}_\mathcal{S})$ to arbitrarily exceed 1.5. Also, it does not assume any particular distribution on entries of $\mathbf{A}$.

Proposition 3. *Let $\mathcal{S}$ be a size- k subset. Assume $k \geq 2$. Let $\mathbf{A}_\mathcal{S}$ be an $k \times n$ random matrix. Let $\delta_{\min}, \delta_{\max}$ be some positive constants. For any sign vector $\boldsymbol{\beta}$ in $\{-1, 1\}^k$, we have*

$$\Pr \left\{ \|(\mathbf{A}_\mathcal{S}^T \mathbf{A}_\mathcal{S})^\dagger \boldsymbol{\beta}\|_\infty > \frac{(\sqrt{k} + 1) \cdot |\tau_k - 1|}{\delta_{\min}^2 \cdot (\tau_k + 1)} \right\} \leq \Pr \{ \mathcal{E}_c(\delta_{\min}, \delta_{\max}) \} \quad (19)$$

where $\mathcal{E}(\delta_{\min}, \delta_{\max}) = \{ \delta_{\min} \leq \sigma_{\min}(\mathbf{A}_\mathcal{S}) \leq \sigma_{\max}(\mathbf{A}_\mathcal{S}) \leq \delta_{\max} \}$, and $\mathcal{E}_c(\delta_{\min}, \delta_{\max})$ is the complementary event of $\mathcal{E}(\delta_{\min}, \delta_{\max})$, and the constant τ_k satisfies

$$\tau_k = \tau_k(\delta_{\max}, \delta_{\min}) = \left(\frac{\delta_{\max}}{\delta_{\min}} \right)^2 \cdot \frac{1 + k^{-\frac{1}{2}}}{1 - k^{-\frac{1}{2}}}. \quad (20)$$

See Appendix C for proof. If $\mathbf{A}_S^T \mathbf{A}_S$ is “almost” an identity matrix, then we expect $\|(\mathbf{A}_S^T \mathbf{A}_S)^{-1} \boldsymbol{\beta}\|_\infty \approx 1$ for any sign vector $\boldsymbol{\beta}$ (hence our above heuristic whereby we set $a_3 = 1$). Proposition 3 makes a slightly weaker (but relatively general) statement. Now for some appropriately fixed $\delta_{\max}$ and $\delta_{\min}$, we expect $\Pr\{\mathcal{E}_c(\delta_{\min}, \delta_{\max})\}$ in (19) to drop exponentially in m . Just as the term $\Pr\{\sigma_{\min}(\mathbf{A}_S) \leq \delta_{\min}\}$ in Proposition 2 can be bounded by $\exp(-m \cdot (0.29 - \delta_{\min})^2 / c_1)$, we can bound⁷ $\Pr\{\sigma_{\max}(\mathbf{A}) > \delta_{\max}\} \leq \exp(-m(\delta_{\max} - 1.71)^2 / c_1)$ for some $\delta_{\max} \geq 1.71$. Roughly speaking, $\sigma_{\max}(\mathbf{A}_S) \leq 1.71$ (or $\sigma_{\max}^2(\mathbf{A}_S) \leq 2.92$) with “high probability”. We fix $\delta_{\min} = a_1$, where a_1 belongs to the *invertability* condition.

So to bound $p(a_3)$, both (18) and Proposition 3 imply $p(a_3) \leq \Pr\{\mathcal{E}_c(\delta_{\min}, \delta_{\max})\}$ for $a_3 = (\sqrt{k} + 1) \cdot |\tau_k - 1| / (\delta_{\min}^2 \cdot (\tau_k + 1))$. Now $\Pr\{\mathcal{E}_c(\delta_{\min}, \delta_{\max})\} \leq 2 \exp(-m \cdot t^2 / c_1)$, where we set $t = \delta_{\max} - 1.71 = 0.29 - a_1$ and $\delta_{\min} = a_1$. By (16), the rate (17) suffices to ensure $u_3 = U_n(a_3) \leq u$ for some fraction u , with the same const. Thus we proved the other main theorem, similar to Theorem 2.

Theorem 3. *Let Φ be an $m \times n$ matrix, where assume n sufficiently large for Theorem 1 to hold. Sample $\Phi = \mathbf{A}$ whereby the entries A_{ij} are IID, and are Gaussian or bounded (as stated in Proposition 2). Then all three invertability, small projections, and invertability projections conditions in LASSO Theorem C for $(\boldsymbol{\beta}, \mathcal{S})$ with $|\mathcal{S}| = k \geq 2$, with $a_1 \leq 0.29$, with a_2 satisfying (7) for some a set in the regularizer θ_n , and with $a_3 = (\sqrt{k} + 1) \cdot |\tau_k - 1| / (a_1^2 \cdot (\tau_k + 1))$ for $\tau_k = \tau_k(1.42 - a_1, a_1)$ in (20), are satisfied for $u_1 + u_2 + u_3 = 4u$ for some small fraction u , if m is on the order of (17) with $\text{const} = \max(4/(a_1 a_2)^2, 2c_1/(0.29 - a_1)^2)$, and c_1 depends on the distribution of A_{ij} ’s. Note $\text{const} \geq 4$.*

In the noiseless limit where only the first 2 conditions are required, this improves to $u_1 + u_2 = 3u$.

Remark 3. *We emphasize that rate (17) is measured w.r.t. to the three conditions in Theorem 3. The probability for which both noise conditions i) and ii) are satisfied, and for which the condition (8) imposed on $\min_{i \in \mathcal{S}} |\alpha_i|$ is satisfied, require additional consideration. For the former the probability is at least $1 - n^{-1}(2\pi \log n)^{-\frac{1}{2}}$, see [6]. We derive the latter using signal statistics, e.g., for $|\alpha_i| \in \mathbb{R}_{[0,1]}$ then $(\min_{i \in \mathcal{S}} |\alpha_i|) > t$ is observed with probability $(1 - t)^k$ with $|\mathcal{S}| = k$.*

Note that the choice for a_3 in Theorem 3 implies $\|(\mathbf{A}_S^T \mathbf{A}_S)^{\dagger} \boldsymbol{\beta}\|_\infty$ is roughly on the order $\sqrt{k}$. Indeed this is true since $\tau_k \geq 1$, and we note $\tau_k = (\delta_{\max}/\delta_{\min})^2 + 2k^{-\frac{1}{2}} + o(k^{-\frac{1}{2}})$, thus $\tau_k \approx (\delta_{\max}/\delta_{\min})^2$ for moderate k . Now LASSO recovery also depends on the probability that condition (8) holds. Our choice for a_3 causes the RHS of (8) to be roughly of the order $c_Z \sqrt{2k \log n}$. Compare this to [6] (see Theorem

⁷For $m \geq 2k$ we have $\Pr\{\sigma_{\max}(\mathbf{A}) > 1.71 + t\} \leq \Pr\{\sigma_{\max}(\mathbf{A}) > 1 + \sqrt{k/m} + t\} \leq \exp(-mt^2/c_1)$ for some c_1 , see [26], Theorem 5.39.

1.3) where it was assumed that $\sigma_{\min}(\mathbf{A}_S) \leq 1.5$, they only require $a_3 = 3$, *viz.*, a factor of $\sqrt{k}$ is lost without this assumption (which was previously argued to be fairly restrictive). To improve Proposition 3, one might assume some specific distributions on $\mathbf{A}$. We leave further improvements to future work.

V. CONCLUSION

We take a first look at U-statistical theory for predicting the average-case behavior of salient CS matrix parameters. Leveraging on the generality of this theory, we consider two different recovery algorithms i) ℓ_1 -minimization and ii) LASSO. The developed analysis is observed to have good potential for predicting CS recovery, and compares well (empirically) with Donoho & Tanner [22] recent average-case analysis for system sizes found in implementations. Measurement rates that incorporate fractional u failure rates, are derived to be on the order of $k[\log((n-k)/u) + \sqrt{2(k/n)\log(n/k)}]$, similar to the known optimal $k \log(n/k)$ rate. Empirical observations suggest possible improvement for const (as opposed to typical worst-case analyses whereby implicit constants are known to be inherently large).

There are multiple directions for future work. Firstly while restrictive maximum eigenvalue assumptions are avoided (as StRIP-recovery does not require them), the applied techniques could be fine-tuned. It is desirable to overcome the $\sqrt{k}$ losses observed here for noisy conditions. Secondly, it is interesting to further leverage generality of U-statistics to other different recovery algorithms, to obtain their average-case analyses. Finally, one might extend the results here for the case where the sampling matrix columns are dependent, which requires appropriate extensions of Theorem 1.

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APPENDIX

A. Proof of Theorem 1

For notational simplicity we shall henceforth drop explicit dependence on a from all three quantities $U_n(a)$, $p(a)$ and $g(\mathbf{A}, a)$ in this appendix subsection. While U_n is made explicit in Definition 1 as a statistic corresponding to the realization $\Phi = \mathbf{A}$, this proof considers U_n consisting of random terms $g(\mathbf{A}_S)$ for purposes of making probabilistic estimates. Theorem 1 is really a *law of large numbers* result. However even when the columns $\mathbf{A}_i$ are assumed to be IID, the terms $g(\mathbf{A}_S)$ in U_n depend on each other. As such, the usual techniques for IID sequences do not apply. Aside from large deviation results such

as Thm. 1, there exist *strong law* results, see [39]. The following proof is obtained by combining ideas taken from [33] and [34]. We use the following new notation just in this subsection of the appendix. Partition the index set $\{1, 2, \dots, n\}$ into $\omega_n = \lfloor n/k \rfloor$ subsets denoted $\mathcal{S}_i$ each of size k , and a single subset $\mathcal{R}$ of size at most k . More specifically, let $\mathcal{S}_i = \{(i-1) \cdot k + 1, (i-1) \cdot k + 2, \dots, i \cdot k\}$ and let $\mathcal{R} = \{\lfloor n/k \rfloor \cdot k + 1, \lfloor n/k \rfloor \cdot k + 2, \dots, n\}$. Let π denote a *permutation* (bijective) mapping $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$. The notation $\pi(\mathcal{S})$ denotes the set of all *images* of each element in $\mathcal{S}$, under the mapping π . Following Section 5c in [33] we express the U-statistic U_n of $\mathbf{A}$ in the form

$$U_n = \frac{1}{n!} \sum_{\pi} \left(\frac{1}{\omega_n} \sum_{i=1}^{\omega_n} g(\mathbf{A}_{\pi(\mathcal{S}_i)}) \right), \quad (21)$$

the first summation taken over all $n!$ possible permutations π of $\{1, 2, \dots, n\}$. To verify, observe that any subset $\mathcal{S}$ is counted exactly $\omega_n \cdot k!(n-k)!$ times in the RHS of (21).

Recall $p = \mathbb{E}g(\mathbf{A}_{\mathcal{S}}) = \mathbb{E}U_n$. From the theorem statement let the term ϵ_n^2 equal $cp(1-p) \cdot \omega_n^{-1} \log \omega_n$ where $c > 2$. We show that the probabilities $\Pr\{|U_n - p| > \epsilon_n\}$ for each n are small. For brevity, we shall only explicitly treat the upper tail probability $\Pr\{U_n - p > \epsilon_n\}$, where standard modifications of the below arguments will address the lower tail probability $\Pr\{-U_n + p > \epsilon_n\}$ (see comment in p. 1, [33]). Using the expression (21) for U_n , write the probability $\Pr\{U_n - p > \epsilon_n\}$ for any $h > 0$ as

$$\Pr\{U_n - p > \epsilon_n\} \leq \mathbb{E} \exp(h(U_n - p + \epsilon_n)) = \mathbb{E} \exp \left(\frac{1}{n!} \left(\sum_{\pi} h(S_{\pi} - p + \epsilon_n) \right) \right),$$

where here S_{π} is a RV that equals the inner summation in (21), i.e. $S_{\pi} = \frac{1}{\omega_n} \sum_{i=1}^{\omega_n} g(\mathbf{A}_{\pi(\mathcal{S}_i)})$. Using convexity of the function $\exp(\cdot)$ we express

$$\Pr\{U_n - p > \epsilon_n\} \leq \frac{1}{n!} \sum_{\pi} \mathbb{E} \exp(h(S_{\pi} - p + \epsilon_n)).$$

Now observe that the RV S_{π} is an average of ω_n IID terms $g(\mathbf{A}_{\pi(\mathcal{S}_i)})$. This is due to the assumption that the columns $\mathbf{A}_i$ of $\mathbf{A}$ are IID, and also due to the fact that the sets $\pi(\mathcal{S}_i)$ are disjoint (recall sets $\mathcal{S}_i$ are disjoint). Hence for any permutation π , by this independence we have $\mathbb{E} \exp(h S_{\pi}) = (\mathbb{E} \exp(h' \cdot g(\mathbf{A}_{\pi(\mathcal{S}_1)})))^{\omega_n}$, where the normalization $h' = h/\omega_n$ bears no consequence. The RV $g(\mathbf{A}_{\pi(\mathcal{S}_1)})$ is bounded, i.e. $0 \leq g(\mathbf{A}_{\pi(\mathcal{S}_1)}) \leq 1$, and its expectation $\mathbb{E}g(\mathbf{A}_{\pi(\mathcal{S}_1)})$ equals p . By convexity of $\exp(\cdot)$ again and for all $h > 0$, the inequality $e^{h\alpha} \leq e^h \alpha + 1 - \alpha$ holds for all $0 \leq \alpha \leq 1$. Therefore putting $\alpha = g(\mathbf{A}_{\pi(\mathcal{S}_1)})$ we get the inequality $\exp(h \cdot g(\mathbf{A}_{\pi(\mathcal{S}_1)})) \leq 1 + (e^h - 1) \cdot g(\mathbf{A}_{\pi(\mathcal{S}_1)})$. By the irrelevance of π in previous arguments, by putting $\mathbb{E}g(\mathbf{A}_{\pi(\mathcal{S}_1)}) = p$

$$\Pr\{U_n - p > \epsilon_n\} \leq e^{-h(\epsilon_n + p)} \left(1 - p + pe^h \right)^{\omega_n}.$$

We optimize the bound by putting $pe^h = (1-p)(p+\epsilon_n)/(1-p-\epsilon_n)$, see (4.7) in [33], to get (note $0 \leq \epsilon_n \leq 1-p$)

$$\Pr\{U_n - p > \epsilon_n\} \leq ((1 + \epsilon_n p^{-1})^{p+\epsilon_n} (1 - \epsilon_n(1-p)^{-1})^{1-p-\epsilon_n})^{-\omega_n}. \quad (22)$$

Following (2.20) in [34] we use the relation $\log(1+\alpha) = \alpha - \frac{1}{2}\alpha^2 + o(\alpha^2)$ as $\alpha \rightarrow 0$, to express the logarithmic exponent on the RHS of (22) as

$$\frac{-\omega_n \epsilon_n^2 \cdot (1 + o(1))}{2p(1-p)}.$$

Therefore by the form $\epsilon_n^2 = cp(1-p) \cdot \omega_n^{-1} \log \omega_n$ where $c > 2$, for sufficiently large n we have

$$\Pr\{U_n - p > \epsilon_n\} \leq \omega_n^{-c/2} < \omega_n^{-1}$$

which in turn implies $\sum_{n=k}^{\infty} \Pr\{U_n - p > \epsilon_n\} < \infty$. Repeating similar arguments for the lower tail probability $\Pr\{-U_n + p > \epsilon_n\}$, we eventually prove $\sum_{n=k}^{\infty} \Pr\{|U_n - p| > \epsilon_n\} < \infty$ which implies the claim.

B. Proof of Proposition 2

For $\tau \in \mathbb{R}$, let $\mathcal{E}(\tau) = \{\beta^T (\mathbf{A}_S^T \mathbf{A}_S)^\dagger \beta \leq \tau\}$ where $\mathcal{E}(\tau)$ is a probabilistic event. Let $\mathcal{E}_c(\tau)$ denote the complementary event. Bound the probability as

$$\Pr \left\{ \left| (\mathbf{A}_S^\dagger \mathbf{A}_\omega)^T \beta \right| > a \right\} \leq \Pr \left\{ \left| (\mathbf{A}_S^\dagger \mathbf{A}_\omega)^T \beta \right| > a \mid \mathcal{E}(\tau) \right\} + \Pr\{\mathcal{E}_c(\tau)\}. \quad (23)$$

We upper bound the first term as follows. Denote constants $c_1, c_2, \dots, c_m$. For entries $(\mathbf{A}_\omega)_i$ of $\mathbf{A}_\omega$, consider the sum $\sum_{i=1}^m c_i \cdot (m^{-\frac{1}{2}} \mathbf{A}_\omega)_i = \frac{1}{m} \sum_{i=1}^m c_i X_i$, where RVs X_i satisfy $X_i = (\sqrt{m} \mathbf{A}_\omega)_i$. By standard arguments we have the double-sided bound $\Pr \{ |\sum_{i=1}^m c_i X_i| > mt \} \leq 2 \exp(-(mt)^2/(2 \cdot \|\mathbf{c}\|_2^2))$, where vector $\mathbf{c}$ equals $[c_1, c_2, \dots, c_m]^T$.

Next write $(\mathbf{A}_S^\dagger \mathbf{A}_\omega)^T \beta = (\sqrt{m} \cdot \beta^T \mathbf{A}_S^\dagger)(m^{-\frac{1}{2}} \mathbf{A}_\omega)$. When conditioning on $\beta^T \mathbf{A}_S^\dagger$, then $\sqrt{m} \cdot \beta^T \mathbf{A}_S^\dagger$ is fixed, say equals some vector $\mathbf{c}$. Put $X_i = (\sqrt{m} \mathbf{A}_\omega)_i$ and X_i 's are independent (by assumed independence of the rows of $\mathbf{A}$). Then use the above bound for $\Pr \{ \sum_{i=1}^m c_i X_i > t \}$, set $t = a$ and conclude

$$\Pr \left\{ \left| (\mathbf{A}_S^\dagger \mathbf{A}_\omega)^T \beta \right| > a \mid \beta^T \mathbf{A}_S^\dagger \right\} \leq 2 \exp \left(-\frac{(ma)^2}{2m \|\beta^T \mathbf{A}_S^\dagger\|_2^2} \right) = 2 \exp \left(-\frac{ma^2}{2 \cdot \beta^T (\mathbf{A}_S^T \mathbf{A}_S)^\dagger \beta} \right),$$

where the last equality follows from the identity $\mathbf{A}_S^\dagger (\mathbf{A}_S^\dagger)^T = (\mathbf{A}_S^T \mathbf{A}_S)^\dagger$. Further conclude that the first term in (23) is bounded by $2 \exp(-ma^2/(2\tau))$, due to further conditioning on the event $\mathcal{E}(\tau) = \{\beta^T (\mathbf{A}_S^T \mathbf{A}_S)^\dagger \beta \leq \tau\}$.

To bound the second term, let $\varsigma_{\max}(\mathbf{A})$ denote the maximum eigenvalue of matrix $\mathbf{A}$. Since $\mathbf{A}_S^T \mathbf{A}_S$ is positive semidefinite, note that $\boldsymbol{\beta}^T (\mathbf{A}_S^T \mathbf{A}_S)^\dagger \boldsymbol{\beta}$ is upper bounded by $\|\boldsymbol{\beta}\|_2^2 \cdot \varsigma_{\max}((\mathbf{A}_S^T \mathbf{A}_S)^\dagger)$, which equals $k \cdot \varsigma_{\max}((\mathbf{A}_S^T \mathbf{A}_S)^\dagger)$. Furthermore $\varsigma_{\max}((\mathbf{A}_S^T \mathbf{A}_S)^\dagger) \leq 1/\sigma_{\min}^2(\mathbf{A}_S)$, where here $\sigma_{\min}(\mathbf{A})$ is the minimum singular value of $\mathbf{A}$. Thus $\Pr\{\mathcal{E}_c(\tau)\} \leq \Pr\{k/\sigma_{\min}^2(\mathbf{A}_S) > \tau\}$. Finally put $\tau = \delta k$ to get $\Pr\{\mathcal{E}_c(\tau)\} \leq \Pr\{\sigma_{\min}^2(\mathbf{A}_S) \leq \delta^{-1}\}$.

C. Proof of Proposition 3

For notational convenience, put $\mathbf{X} = (\mathbf{A}_S^T \mathbf{A}_S)^\dagger$. Bound the probability

$$\Pr\left\{\|\mathbf{X}\boldsymbol{\beta}\|_\infty > a\sqrt{k}\right\} \leq \Pr\left\{\|\mathbf{X}\boldsymbol{\beta}\|_\infty > a\sqrt{k} \mid \mathcal{E}(\delta_{\min}, \delta_{\max})\right\} + \Pr\{\mathcal{E}_c(\delta_{\min}, \delta_{\max})\}. \quad (24)$$

where we take a to mean

$$a = \frac{|\tau_k - 1|}{\tau_k + 1} \cdot \frac{1 + k^{-\frac{1}{2}}}{\sigma_{\min}^2(\mathbf{A}_S)} \quad (25)$$

for τ_k chosen as in (20). We claim that every entry $(\mathbf{X}\boldsymbol{\beta})_i$ of $\mathbf{X}\boldsymbol{\beta}$ is upper bounded by $a\sqrt{k}$, for a as in (25). Then by definition of $\mathcal{E}(\delta_{\min}, \delta_{\max})$, the first term in (24) equals 0 and we would have proven the bound (19).

Let $\mathbf{C}$ denote a $k \times 2$ matrix. The first column $\mathbf{C}$ is be a normalized version of $\boldsymbol{\beta}$, more specifically it equals $k^{-\frac{1}{2}}\boldsymbol{\beta}_i$. The second column equals the canonical basis vector $\mathbf{c}_i$, where $\mathbf{c}_i$ is a 0-1 vector whereby $(\mathbf{c}_i)_j = 1$ if and only if $j = i$. Consider the 2×2 matrix $\mathbf{X}'$ that satisfies $\mathbf{X}' = \mathbf{C}^T \mathbf{X} \mathbf{C}$. This matrix $\mathbf{X}'$ is symmetric (from symmetry of $\mathbf{X}$) and $k^{-\frac{1}{2}}(\mathbf{X}\boldsymbol{\beta})_i = X'_{1,2} = X'_{2,1}$ (from our construction of $\mathbf{C}$). That is the entry $X'_{1,2}$ (and $X'_{2,1}$) of $\mathbf{X}'$, correspond to the (scaled) quantity $k^{-\frac{1}{2}}(\mathbf{X}\boldsymbol{\beta})_i$ that we want to bound.

Condition on the event $\mathcal{E}_c(\delta_{\min}, \delta_{\max})$, then $\mathbf{A}_S$ has rank k and therefore $\mathbf{X} = (\mathbf{A}_S^T \mathbf{A}_S)^\dagger = (\mathbf{A}_S^T \mathbf{A}_S)^{-1}$. Let $\det(\cdot)$ and $\text{Tr}(\cdot)$ denote determinant and trace. As in [38] equation (11), we have

$$1 - \frac{X'_{1,2}X'_{1,2}}{X'_{1,1}X'_{2,2}} = \frac{4 \det(\mathbf{X}')}{(\text{Tr}(\mathbf{X}'))^2 - (X'_{1,1} - X'_{2,2})^2} \geq \frac{4\varsigma_{\max}(\mathbf{X}') \cdot \varsigma_{\min}(\mathbf{X}')}{(\text{Tr}(\mathbf{X}'))^2} = \frac{4t}{(1+t)^2} \quad (26)$$

where $t = \varsigma_{\max}(\mathbf{X}')/\varsigma_{\min}(\mathbf{X}')$ and $\varsigma_{\max}$ and $\varsigma_{\min}$ respectively denote the maximum and minimum eigenvalues. Now $t = \varsigma_{\max}(\mathbf{X}')/\varsigma_{\min}(\mathbf{X}') \geq 1$. If $t = 1$ then $4t/(1+t)^2 = 1$, and for $t \geq 1$ the function $4t/(1+t)^2$ decreases monotonically. We claim that τ_k in (20) upper bounds $t = \varsigma_{\max}(\mathbf{X}')/\varsigma_{\min}(\mathbf{X}')$, and (26) then allows us to produce the following upper bound

$$|X'_{1,2}| \leq \sqrt{X'_{1,1}X'_{2,2} \cdot \left(1 - \frac{4\tau_k}{(1+\tau_k)^2}\right)} = \sqrt{X'_{1,1}X'_{2,2}} \cdot \frac{|\tau_k - 1|}{1 + \tau_k}.$$

Bound $(X'_{1,1}X'_{2,2})^{\frac{1}{2}}$ by the maximum eigenvalue $\varsigma_{\max}(\mathbf{X}')$ of $\mathbf{X}'$. Then, further bound $\varsigma_{\max}(\mathbf{X}')$ by $(1 + k^{-\frac{1}{2}})/\sigma_{\min}^2(\mathbf{A}_S)$, which gives the form (25). This bound is argued as follows. For $k \geq 2$, we have the columns in $\mathbf{C}$ to be *linearly independent*. Since $\mathbf{X}' = \mathbf{C}^T \mathbf{X} \mathbf{C}$ and $\mathbf{X}$ is *positive definite*, it is then clear that $\varsigma_{\max}(\mathbf{X}') \leq \varsigma_{\max}(\mathbf{C}^T \mathbf{C}) \cdot \varsigma_{\max}(\mathbf{X})$. Now $\mathbf{C}^T \mathbf{C}$ is a 2×2 matrix with diagonal elements 1, and off-diagonal elements $\pm 1/\sqrt{k}$. Hence $\varsigma_{\max}(\mathbf{C}^T \mathbf{C}) = 1 + k^{-\frac{1}{2}}$. Also $\varsigma_{\max}(\mathbf{X}) \leq 1/\sigma_{\min}^2(\mathbf{A}_S)$, and the bound follows.

To finish, we show the claim $\tau_k \geq \varsigma_{\max}(\mathbf{X}')/\varsigma_{\min}(\mathbf{X}')$. By similar arguments as above, it follows that

$$\frac{\varsigma_{\max}(\mathbf{X}')}{\varsigma_{\min}(\mathbf{X}')} \leq \frac{\varsigma_{\max}(\mathbf{C}^T \mathbf{C})}{\varsigma_{\min}(\mathbf{C}^T \mathbf{C})} \cdot \frac{\varsigma_{\max}(\mathbf{X})}{\varsigma_{\min}(\mathbf{X})} = \frac{1 + k^{-\frac{1}{2}}}{1 - k^{-\frac{1}{2}}} \cdot \frac{\sigma_{\max}^2(\mathbf{A}_S)}{\sigma_{\min}^2(\mathbf{A}_S)} \leq \tau_k$$

since $\varsigma_{\min}(\mathbf{X}') \geq \varsigma_{\min}(\mathbf{C}^T \mathbf{C}) \cdot \varsigma_{\min}(\mathbf{X})$, and $\varsigma_{\min}(\mathbf{X}') = 1 - k^{-\frac{1}{2}}$, and $\mathbf{X} = (\mathbf{A}_S^T \mathbf{A}_S)^{-1}$. We are done.

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