Abstract—Compressed Sensing (CS) is a signal acquisition approach aiming to reduce the number of measurements required to capture a sparse (or, more generally, compressible) signal. Several works have shown significant performance advantages over conventional sampling techniques, through both theoretical analyses and experimental results, and have established CS as an efficient way to acquire and reconstruct a sparse signal with low sampling rate, even below Nyquist. However, apart from the full signal recovery, in some cases only certain features and properties of the signal are required. For instance, in many wireless communication applications, e.g. cognitive radio systems and channel estimation, the indices recovery of the non-zero components of a sparse signal is required. In this work, we study the problem of sparse support recovery from few and noisy signal measurements by examining both fundamental bounds and performance of some of the prevailing practical algorithms for support recovery. After summarizing some algorithm-independent information theoretic limits, we present four frequently used support recovery algorithms with different complexity requirements and recovery capabilities: (i) maximum correlation (MC) algorithm, (ii) a thresholded greedy optimization algorithm, (iii) an $\ell_1$-constrained quadratic programming approach, and (iv) thresholded basis pursuit (TBP) algorithm. Their performance is simulated and examined for different sparsity levels, number of measurements, signal characteristics (i.e. $\beta_{min, MA}$) and $SNR$ values. Our simulation results indicate that TBP outperforms the other methods at the expense of higher complexity. In addition, simulation results emphasize the importance of signal characteristics on the success of the support recovery.

Index Terms—sparse support recovery; sparsity pattern; information theoretic limits; practical algorithms for support recovery

I. INTRODUCTION

Compressed Sensing (CS) has been an emerging research field the last decade, mainly because of the development of fast algorithms for sparse signal recovery, accompanied with strong guarantees about their expected performance [1], [2]. Several convex relaxations (i.e. basis pursuit [3], Dantzig selector [4]) and greedy algorithms (i.e. matching pursuit and its variations [5], [6], iterative thresholding methods [7], [8]) have been proposed in the literature with astonishing performance, verified both through theoretical analyses and experimental verifications.

Among the CS-related research topics, the study of fundamental limits of sparse signal recovery has been of great interest, especially in the presence of noise. Some information-theoretic approaches have been proposed in the literature, trying to characterize necessary and sufficient condition, as well as to derive bounds relating the number of measurements, the sparsity level, the signal-to-noise ratio ($SNR$) and the recovering distortion. A representative attempt in this direction proposed in [9], models the measurement process as passing noise-free measurements through a noisy stochastic channel, deriving a lower bound on the rate required to achieve a certain reconstruction distortion.

However, apart from signal recovery based on few noisy measurements, an interesting question is the ability to accurately recover specific signal properties or characteristics only. One property of great importance and practical value for sparse signals is the signal support, or alternatively sparsity pattern, which corresponds to the set of indices of non-zero components of the signal. Several works have studied sparsity pattern recovery, providing sufficient and necessary conditions on the number of measurements and required $SNR$, mainly for exact support recovery [10], [11], [12]. In addition, approximate support recovery has been addressed in [12], [13], examining the trade-off between number of measurements and detection errors.

Recovery of sparse signals’ support is a problem encountered quite often in many sciences and engineering fields. For instance, in gene regulatory networks, identifying the genes related the most with the onset of a disease or some of its symptoms can be formulated as a sparse support recovery problem [14], mainly because, in most cases, the number of genes highly correlated with a disease is extremely low. In the wireless communications field, sparse support recovery has found several applications so far. In cognitive radio systems primary users (PUs) are eligible to request and reserve the available spectrum depending on their demands, while secondary users (SUs) are only allowed to use the communications medium if it is not utilized by PUs. According to this approach, SUs can opportunistically and independently from PUs discover the unused spectrum bins and decide on their transmission frequency band after modeling the spectrum sensing as a sparse support problem [15]. In addition, sparse support recovery has been employed as an efficient method for channel estimation since, in general, there are few distinct signal arrivals than baseband channel taps [16], [17].

In this work, we consider the problem of sparse support recovery from few and noise-corrupted measurements. After presenting some representative results on fundamental bounds for exact and approximate sparse support recovery and providing intuition on the main asymptotic results, we study practical recovery algorithms for support recovery of different
computational requirements and expected performance. In more detail, we examine four recovery algorithms: (i) a simple maximum correlation (MC) approach which acquires directly the signal support with an one-time operation [11], (ii) a greedy signal recovery algorithm, based on CoSaMP, followed by a thresholding operator [8], (iii) an $\ell_1$-constrained quadratic programming approach, commonly referred to as least absolute shrinkage and selection operator (LASSO) [18], and (iv) an optimized support recovery algorithm named thresholded basis pursuit (TBP) [19].

The rest of the paper is organized as follows. Section II defines the notation, main definitions and metrics used across the paper, while Section III presents some of the information theoretic results on support recovery. In Section IV, the four recovery algorithms are described and their performance simulation results are presented and compared. Finally, Section V concludes the paper.

II. BACKGROUND

Let $x \in \mathbb{R}^n$ be a $k$-sparse signal. The CS encoding process for signal $x$ is

$$y = Ax,$$  \hspace{1cm} (1)

where $A \in \mathbb{R}^{m \times n}$ is called measurement matrix. If $m \sim O(k \log (n/k))$ measurements are taken from the signal $x$, CS can achieve exact recovery with high probability [1] if $A$ satisfies the RIP condition

$$(1 - \delta_k)\|x\|_2 \leq \|Ax\| \leq (1 + \delta_k)\|x\|_2,$$  \hspace{1cm} (2)

for all $k$-sparse signals $x$, where $\delta_k$ is a constant $\in (0,1)$. In the case of transmitting the measurements through a noisy channel $y$ can be written as

$$y = Ax + z,$$  \hspace{1cm} (3)

where $z \sim \mathcal{N}(0, \sigma^2 I_{m \times m})$. The signal reconstruction is mainly performed through an optimization problem, which is usually referred to as basis pursuit (BP) and can be formulated as

$$\min_{\hat{x} \in \mathbb{R}^n} \|\hat{x}\|_{\ell_1}, \text{ s. t. } y = Ax.$$  \hspace{1cm} (4)

Apart from the signal itself, quite often some of its features are required, one of them being its support. Assume

$$S = S(x) := \{i \in \{1,2,\ldots,n\}|x_i \neq 0\}$$  \hspace{1cm} (5)

represent the support of signal $x$. Then, support recovery is the mapping

$$(y, A) \in \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow$$

$$T := \{T|T \subseteq \{1,2,\ldots,m\}, |T| = k\}. $$  \hspace{1cm} (6)

If $\psi(\cdot)$ is a general support recovery decoder implementing a hypothetical algorithm, then

$$\hat{S} = \psi(y, A),$$  \hspace{1cm} (7)

where $\hat{S}$ is the recovered signal support.

Several error metrics have been introduced in the literature to capture the effect of the erroneous support reconstruction. For exact recovery, the most popular metric is the average error probability of $\psi(\cdot)$

$$q_{\text{avg}}(\psi) := \frac{1}{|S|} \sum_{S \in T_k} \mathbb{P}[\hat{S} \neq S|S],$$  \hspace{1cm} (8)

or the maximum error probability

$$q_{\text{max}}(\psi) := \max_{S \in T_k} \mathbb{P}[\hat{S} \neq S|S],$$  \hspace{1cm} (9)

which corresponds to adversarially choosing $S$. An other quite often error metric, especially used in approximate support recovery, is the error probability defined as

$$d(S, \hat{S}) = \max \left( \frac{1}{|S|} \sum_{i \in S} 1(i \in S, i \notin \hat{S}) \right),$$  \hspace{1cm} (10)

$$d(S, \hat{S})$$ captures the number of missed-detection and false alarms.

III. INFORMATION THEORETIC LIMITS FOR SPARSE SUPPORT RECOVERY

It can be shown that it is impossible to identify the support of a continuous signal $x$ from noisy measurements with arbitrary small non-zero components. For this reason, a significant parameter of the support recovery problem is the magnitude of the smallest non-zero component of $x$ ($\beta_{\min}$), which should be bounded away from zero. An other important parameter introduced in [11] is the minimum-to-average ratio

$$\text{MAR}(x) = \frac{\beta_{\min}^2}{\|x\|_2^2/k},$$  \hspace{1cm} (11)

which characterizes the spread of the non-zero components of $x$. It can be easily shown that $\text{MAR} \in (0,1]$. The effect of $\beta_{\min}$ and $\text{MAR}$ on information theoretic support recovery bounds is captured in this Section, while next Section examines their effect on the performance of practical recovery algorithms.

Several authors have examined the problem of deriving necessary and sufficient conditions for exact sparse support recovery. If $A$ is assumed to be Gaussian ($A = \{a_{ij}|a_{ij} \sim \mathcal{N}(0, \sigma^2)\}$) and an optimal decoder ($\psi_{\text{opt}}$) is considered, it is proved [10] that if

$$m > c_1 \max\{k \log (n/k), \frac{\log (n-k)}{\sigma \sqrt{n}}\},$$  \hspace{1cm} (12)

then

$$q_{\text{avg}}(\psi_{\text{opt}}) \leq \exp(-c_2(m-k)),$$  \hspace{1cm} (13)

where $c_1$ and $c_2$ constants. This means that the error probability can be made arbitrarily small ensuring exact recovery. A similar argument can be made for the worst-case error probability but in that case an extra penalty in the number of measurements is required. In more detail, if

$$m > \log \left( \frac{n}{k} \right) + c_1 \max\{k \log (n/k), \frac{\log (n-k)}{\sigma \sqrt{n}}\},$$  \hspace{1cm} (14)
then
\[ q_{\max}(\psi_{\text{opt}}) \leq \exp(-c_2(m - k)). \] (15)

Apart from sufficient conditions, necessary conditions which must be satisfied by any decoder are derived as well. If
\[ m < \max \left\{ \frac{\log \left( \frac{n}{k} \right)}{8k^2\sigma^2}, \frac{\log n - k}{4\sigma^2\gamma_{\text{min}}} \right\}, \] (16)
then there exists \( x \in \mathbb{U}(\beta_{\text{min}}) \), where \( \mathbb{U}(\beta_{\text{min}}) = \{ x \in \mathbb{R}^n \mid ||x|| \geq \beta_{\text{min}} \text{ for all } i \in S(x) \} \), such that
\[ q_{\max}(\psi) \geq q_{\text{avg}}(\psi) \geq \frac{1}{2}. \] (17)

Based on the mentioned algorithm-independent results on sparse support recovery, some performance bounds on the number of measurements, \( \text{SNR} \) and signal sparsity can be summarized by the following:

- When there is no noise and \( x \) is exactly \( k \)-sparse, then sparse signal recovery through \( \ell_1 \)-minimization can exactly recover the signal, and thus its support, with \( m = O(k \log (n/k)) \) measurements.
- In the noisy case, exact signal recovery is not possible but exact support recovery is. In order to achieve exact support recovery, \( \beta_{\text{min}} \) should be bounded away from zero and \( m = \Theta(k \log (n - k)) \).
- If \( k \leq an \), where \( a \) is a small positive number, and \( m = \Omega(k \log (n/k)) \), then for exact support recovery \( \text{SNR} \) should scale to infinite with \( n \), as \( \text{SNR} = \Omega(\log n) \), which means that \( \text{SNR} \) should go to infinity as \( n \) grows.
- Approximate support recovery within some fixed error can be achieved with finite \( \text{SNR} \) [13].

IV. PRACTICAL SPARSE SUPPORT RECOVERY ALGORITHMS

Several algorithms have been proposed in the literature for sparse support recovery with different complexity requirements and recovery performance. We present the main approaches and compare their achieved recovery capabilities in different scenarios.

- **Maximum correlation (MC) algorithm:** MC is the least complex algorithm among the presented ones and its performance is analyzed in [11]. The algorithm calculates the correlation between the columns of matrix \( A \) and the measurement vector \( y \), and chooses the \( k \) locations with the highest value
\[ S_{\text{MC}} = \{ i : \rho(i) = \frac{|a_i'y|^2}{\|a_i\|^2} \text{ choose } k \text{ largest values} \}. \] (18)

Thus, it is an one-time operation with a sorting process for the \( k \) largest components.

- **Thresholded CoSaMP algorithm:** An approach for sparse support recovery is to initially perform sparse signal reconstruction and then apply a thresholding operator. A greedy signal reconstruction algorithm with good performance, particularly in the presence of noise, is CoSaMP [8]. CoSaMP iteratively creates a signal proxy by least squares estimation, support merging and signal pruning in every iteration until the stopping criterion. Following the signal recovery algorithm, a thresholding operator selects the \( k \) components with the largest magnitude.

- **LASSO algorithm:** An \( \ell_1 \)-constrained quadratic programming approach, known as LASSO, formulates the signal recovery process as the following problem
\[ \min_{x \in \mathbb{R}^n} \| y - A\hat{x} \|_2^2 + \lambda \| \hat{x} \|_1, \] (19)
where \( \lambda \geq 0 \) is an \( \ell_1 \)-regularization parameter. Similarly to the previous approach, the signal recovery algorithm is followed by a thresholding operator for the support selection.

- **Thresholded basis pursuit (TBP):** The TBP algorithm is composed initially by a regular basis pursuit, given by Eq. 4, followed by a thresholding and debasing step [19].

A. Experimental Setup and Simulation Results

We evaluate the performance of the aforementioned sparse support recovery algorithms through a simulation setup. In more detail, we study the average support error probability given by Eq. 9 in terms of the number of measurements \( m \), sparsity level \( k \), \( \text{SNR} \), \( MAR \) and \( \beta_{\text{min}} \). For the purposes of the experiments we generate random sparse signals with 1024 coefficients \( n = 1024 \) of exactly \( k \) sparsity and each experiment is performed and averaged over 20 times. It should be emphasized that, for simplicity purposes, \( \beta_{\text{min}} \) and \( MAR \) serve as lower bounds for the generated signals. This means that a signal \( x \) in our simulations with \( \beta_{\text{min}} \) of 0.1 means that \( \beta_{\text{min}}(x) \geq \beta_{\text{min}} \).

In Fig. 1 the average error probability of the four algorithms with respect to \( MAR \) is shown, while in Fig. 2 the error probability is plotted against \( \beta_{\text{min}} \). It is shown that all algorithms have worse recovery as the minimum signal components reduce and approach the noise perturbation. In these cases, the recovery algorithms occasionally fail to identify the correct support because parts of the signals are indistinguishable from the noise.
Fig. 2: Average support error probability with respect to $\beta_{\text{min}}$. Simulation parameters: $k = 30, m = 300, SNR = 15dB$ and $MAR = 10^{-4}$.

Fig. 3: Average support error probability with respect to $SNR$. Simulation parameters: $k = 30, m = 200, MAR = 10^{-4}$ and $\beta_{\text{min}} = 0.2$.

Fig. 4: Average support error probability with respect to $SNR$. Simulation parameters: $k = 30, m = 350, MAR = 10^{-4}$ and $\beta_{\text{min}} = 0.2$.

Fig. 5: Average support error probability with respect to $M$. Simulation parameters: $k = 15, SNR = 10, MAR = 10^{-4}$ and $\beta_{\text{min}} = 0.2$.

V. CONCLUSION

The problem of recovering the non-zero indices of a sparse signal from few noisy measurements, commonly referred to as sparse support recovery, is considered in this work. The problem of sparse support recovery is encountered in several engineering and science fields, with a particular emphasis in wireless communications. We present some algorithm-independent bounds on the required number of measurements, signal-to-noise ratio and signal properties for exact and approximate support recovery asymptotically. In addition, we describe four prevailing sparse support recovery algorithms and compare their performance in different scenarios. In more detail, a maximum-correlation algorithm (MC), a greedy approach based on CoSaMP, an $\ell_1$-constrained quadratic programming method and a thresholded basis pursuit algorithm (TBP) are examined. Our simulations results suggest that signal characteristics, such as $\beta_{\text{min}}$ and $MAR$, are of major importance for both exact and approximate support recovery, as the fundamental limits predict. Furthermore, the recovery capabilities of the examined algorithms varies accordingly to their computational requirements, with MC and TBP being the...
least and most well performing algorithm, respectively.

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