A Linear Network Code Construction for General Integer Connections Based on the Constraint Satisfaction Problem

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Abstract— The problem of finding network codes for general connections is inherently difficult. Resource minimization for general connections with network coding is further complicated. The existing solutions mainly rely on very restricted classes of network codes, and are almost all centralized. In this paper, we introduce linear network mixing coefficients for code constructions of general connections that generalize random linear network coding (RLNC) for multicast connections. For such code constructions, we pose the problem of cost minimization for the subgraph involved in the coding solution and relate this minimization to a Constraint Satisfaction Problem (CSP) which we show can be simplified to have a moderate number of constraints. While CSPs are NP-complete in general, we present a probabilistic distributed algorithm with almost sure convergence in finite time by applying Communication Free Learning (CFL). Our approach allows fairly general coding across flows, guarantees no greater cost than routing, and shows a possible distributed implementation.

I. INTRODUCTION

The problem of finding network codes in the case of general connections, where each destination can request information from any subset of sources, is intrinsically difficult and little is known about its complexity. In certain special cases, such as multicast connections (where destinations share all of their demands), it suffices to satisfy a Ford-Fulkerson type of min-cut max-flow constraint between all sources to every destination individually. For multicast connections, linear codes suffice [1], [2], and lend themselves to a distributed random construction [3]. While linear codes have been the most widely considered in the literature, linear codes over finite fields may in general not be sufficient for general connections. In addition, even when we consider simple scalar network codes, which have scalar coding coefficients, the problem of code construction for general connections remains vexing. The main difficulty lies in canceling the effect of flows that are coded together even though they are not destined for a common destination.

The problem of code construction is further complicated when we seek, for common reasons of network resource management, to fully or partially limit the use of links in the network. For convex cost functions of flows over edges in the graph corresponding to the network, finding a minimum-cost solution is known to be a convex optimization problem in the case of multicast connections (for continuous flows). However, in the case of general connections, network resource minimization, even when allowing only restricted code constructions, appears difficult.

Among coding approaches for optimizing network use for general connections, we distinguish two types. The first, which we adopt in this paper, is that of mixing, by which we mean coding together flows using RLNC [3], originally proposed for multicast connections. The principle is to code together flows as though they were part of a common multicast connection. In this case, no explicit coding coefficients are provided, and decidability is ensured with high probability by the RLNC. For example, the mixing approaches in [4] and [5] are both based on mixing variables, each corresponding to a set of flows which can be mixed over an edge. Specifically, in [4], a two-step mixing approach is proposed for network resource minimization of general connections, where flow partition (mixing) and flow rate optimization are considered separately. This separation imposes stronger restrictions on the mixing design in the first step and leads to a limitation on the feasibility region. Reference [5] studies the feasibility of more general mixing designs based on mixing variables of size $O(2^P)$, where $P$ is the number of flows. However, [5] does not provide an approach for obtaining a specific mixing design. The second type of coding approach is an explicit linear code construction, by which we mean providing specific linear coefficients over some finite field, to be applied to flows at different nodes. Often these constructions are simplified by restricting them to be binary, i.e., coding flows together only pairwise. For example, in [6] and [7], simple codes over pairs of flows are proposed for network resource minimization of general connections.

Some explicit linear network code construction approaches [6], [7] are distributed, but they allow only pairwise coding. The algorithms of [8] using evolutionary techniques, which are also explicit code constructions, are partially distributed, since the chromosomes can be decomposed into their local contributions, but information has to be fed back from the receivers to all the nodes in the network. In addition, the convergence results for evolutionary techniques are generally scant and do not yield prescriptive constructions.

While RLNC for multicast connections is a distributed
algorithm, most of the mixing approaches [4], [5] based on it have remained centralized. In [9], we propose new methods for constructing linear network codes for general connections of continuous flows based on mixing to minimize the total network cost. Flow splitting and coding over time are required to achieve the desired performance. The focus in [9] is to apply continuous optimization techniques to obtain continuous flow rates.

Our contribution of this paper is to present a new method for constructing linear network codes in a distributed manner for general connections of integer flows based on mixing.

- We introduce linear network mixing coefficients with size polynomial in the number of flows, and formally establish the relationship between linear network coding and mixing.
- We formulate the minimization of the cost of the subgraph involved in the code construction for general connections of integer flows in terms of the mixing coefficients.
- We relate our problem to a CSP [10]. While CSPs are NP-complete in general, we present a probabilistic distributed algorithm with almost sure convergence in finite time by applying CFL, a recent probabilistic distributed solution for CSPs [10].
- We show that our approach guarantees no greater cost than routing.

While our approach, like all other general connection code constructions, is generally suboptimal, it allows more flows to be mixed than is possible with pairwise mixing [6], [7] and with the separate mixing design in [4]. Moreover, different from [4], [5], [9], our approach does not require coding over time.

II. PROBLEM SETUP AND DEFINITIONS

A. Network Model

We consider a directed acyclic network with general connections.\(^1\) Let \(G = (V, E)\) denote the directed acyclic graph, where \(V\) denotes the set of \(V = |V|\) nodes and \(E\) denotes the set of \(E = |E|\) edges. To simplify notation, we assume there is only one edge from node \(i \in V\) to node \(j \in V\), denoted as edge \((i, j) \in E\).\(^2\) For each node \(i \in V\), define the set of incoming neighbors to be \(I_i = \{j : (j, i) \in E\}\) and the set of outgoing neighbors to be \(O_i = \{j : (i, j) \in E\}\). Let \(I_i = |I_i|\) and \(O_i = |O_i|\) denote the in-degree and out-degree of node \(i \in V\), respectively. Assume \(I_i \leq D\) and \(O_i \leq D\) for all \(i \in V\), where \(D\) is a constant. Let \(P = \{1, \cdots, P\}\) denote the set of \(P = |P|\) flows to be carried by the network. For each flow \(p \in P\), let \(s_p \in V\) be its source. We consider integer flows. To simplify notation, we assume unit source rate (i.e., one finite field symbol per second).\(^3\) Let \(S = \{s_1, \cdots, s_P\}\) denote the set of \(P = |S|\) sources. We assume different flows do not share a common source node and no source node has any incoming edges. Let \(T = \{t_1, \cdots, t_T\}\) denote the set of \(T = |T|\) terminals. Each terminal \(t \in T\) demands a subset of \(P_t = |P_t|\) flows \(P_t \subseteq P\). Assume \(U_t \in T P_t = P\). We assume no terminal has any outgoing edges.

As we consider integer flows, we assume unit edge capacity (i.e., one finite field symbol per second).\(^4\) Let \(z_{ij} \in \{0, 1\}\) denote whether edge \((i, j) \in E\) is in the subgraph involved in the code construction.\(^5\) We assume a cost is incurred on an edge when information is transmitted through the edge. Let \(U_{ij}(z_{ij})\) denote the cost function incurred on edge \((i, j)\). Assume \(U_{ij}(z_{ij})\) is non-decreasing in \(z_{ij}\). We are interested in the problem of finding linear network coding designs and minimizing the network cost \(\sum_{(i, j) \in E} U_{ij}(z_{ij})\) for general connections under those designs.

B. Scalar Time-Invariant Linear Network Coding

Consider a finite field \(\mathcal{F}\) with size \(F = |\mathcal{F}|\). In linear network coding, a linear combination over \(\mathcal{F}\) of the symbols in \(\{\sigma_{kj} : \mathcal{F} = \{k \in \mathcal{I}_j\}\}\) from the incoming edges \(\{(k, i) : k \in \mathcal{I}_j\}\) can be transmitted through the shared edge \((i, j) \in E\). The coefficients used to form this linear combination are referred to as local coding coefficients. Specifically, let \(a_{kij} \in \mathcal{F}\) denote the local coding coefficient corresponding to edge \((k, i) \in E\) and edge \((i, j) \in E\). Then, for linear network coding, using local coding coefficients, the symbol through edge \((i, j) \in E\) can be expressed as

\[\sigma_{ij} = \sum_{k \in \mathcal{I}_i} a_{kij}\sigma_{ki}, \quad (i, j) \in E, \quad i \not\in S.\]  \hspace{1cm} (1)

Starting from the sources, we transmit source symbols \(\{\sigma_p \in \mathcal{F} : p \in \mathcal{P}\}\), and then, at intermediate nodes, we perform only linear operations over \(\mathcal{F}\) on the symbols from incoming edges. Thus, the symbol of each edge can be expressed as a linear combination over \(\mathcal{F}\) of the symbols from incoming edges. This is referred to as the global coding coefficient of flow \(p \in P\) and edge \((i, j) \in E\). Let \(c_{ij,p} \in \mathcal{F}\) denote the coefficient of flow \(p \in \mathcal{P}\) in the linear combination for edge \((i, j) \in E\). This is referred to as the global coding vector of edge \((i, j) \in E\).

Here, \(F^P\) represents the set of global coding vectors, the size of which is \(F^P\). Then, using global coding vectors, the symbol through edge \((i, j) \in E\) can also be expressed as

\[\sigma_{ij} = \sum_{p \in \mathcal{P}} c_{ij,p}\sigma_p, \quad (i, j) \in E, \quad i \not\in S.\]  \hspace{1cm} (2)

Note that, in this paper, we consider scalar time-invariant linear network coding. In other words, \(a_{kij} \in \mathcal{F}\) and \(c_{ij,p} \in \mathcal{F}\).

\(^2\)An edge with a positive integer edge capacity greater than one can be equivalently converted to multiple edges, each with unit edge capacity.

\(^3\)There is either no flow or a unit rate of (coded) flow through each edge. Under the unit source rate and edge capacity assumptions, we shall see that there is one global coding (mixing) vector for each edge.
are both scalars, and do not change over time. Let \( e_p \) denote the vector with the \( p \)-th element being 1 and all the other elements being 0. For decodability to hold at all the terminals, the global coding vectors at all edges must satisfy the following definition of feasibility for scalar linear network coding.

**Definition 1 (Feasibility of Scalar Linear Network Coding):** For a network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) and a set of flows \( \mathcal{P} \) with sources \( S \) and terminals \( T \), a linear network code \( \{ c_{ij} \in \mathbb{F}^P : (i, j) \in \mathcal{E} \} \) is called feasible if the following three conditions are satisfied: 1) \( c_{s_p,j} = e_p \) for source edge \( (s_p,j) \in \mathcal{E} \), where \( s_p \in S \) and \( p \in \mathbb{P} \); 2) \( c_{ij} = \sum_{k \in \mathcal{L}} \alpha_{kij} c_{ki} \) for edge \( (i, j) \in \mathcal{E} \) not outgoing from a source, where \( \alpha_{kij} \in \mathbb{F} \); 3) \( e_p \in \text{span}\{c_\ell_i : i \in \mathcal{I}_t\} \), where \( p \in \mathcal{P}_t \) and \( t \in \mathcal{T} \).

Note that when using scalar linear network coding, for each terminal, extraneous flows are allowed to be mixed with the desired flows on the paths to the terminal, as the extraneous flows can be cancelled at intermediate nodes, or at the terminal.

**C. Scalar Time-Invariant Linear Network Mixing**

As mentioned in Section I, to facilitate distributed linear network code designs for general connections using the mixing concept (without requiring the specific values of local or global coding coefficients in the designs), we introduce local and global mixing coefficients. Specifically, we introduce the local mixing coefficient \( \beta_{kij} \in \{0,1\} \) corresponding to edge \((k,i) \in \mathcal{E}\) and edge \((i,j) \in \mathcal{E}\), which relates to the local coding coefficient \( \alpha_{kij} \in \mathbb{F} \). \( \beta_{kij} = 1 \) indicates that symbol \( c_{ki} \) of edge \((k,i) \in \mathcal{E}\) is allowed to be mixed with the local combination over \( \mathcal{F} \) forming symbol \( c_{ij} \) in (1) and \( \beta_{kij} = 0 \) otherwise. Thus, if \( \beta_{kij} = 0 \), we have \( \alpha_{kij} = 0 \) (note that \( \alpha_{kij} \) can be zero when \( \beta_{kij} = 1 \)).

Similarly, we introduce the global mixing coefficient \( x_{i,j,p} \in \{0,1\} \) of flow \( p \in \mathcal{P} \) and edge \((i,j) \in \mathcal{E}\), which relates to the global coding coefficient \( c_{ij,p} \in \mathbb{F} \). \( x_{i,j,p} = 1 \) indicates that flow \( p \) is allowed to be mixed (coded) with other flows, i.e., symbol \( c_{ij} \) is allowed to contribute to the linear combination over \( \mathcal{F} \) forming symbol \( c_{ij} \) in (2), and \( x_{i,j,p} = 0 \) otherwise. Thus, if \( x_{i,j,p} = 0 \), we have \( c_{ij,p} = 0 \) (note that \( c_{ij,p} \) can be zero when \( x_{i,j,p} = 1 \)). Then, we introduce the global mixing vector \( x_{i,j,p} = (x_{ij1}, \ldots, x_{ij,p}, \ldots, x_{ijP}) \in \{0,1\}^P \) for edge \((i,j) \in \mathcal{E}\), which relates to the global coding vector \( c_{ij} = (c_{ij1}, \ldots, c_{ij,p}, \ldots, c_{ijP}) \in \mathbb{F}^P \). Here, \( \{0,1\}^P \) represents the set of global mixing vectors, the size of which is \( 2^P \).

We consider scalar time-invariant linear network mixing. In other words, \( \beta_{kij} \in \{0,1\} \) and \( x_{i,j,p} \in \{0,1\} \) are both scalars, and \( \beta_{kij} \) and \( x_{i,j,p} \) do not change over time.

Global mixing vectors provide a natural way of speaking of flows as possibly coded or not without knowledge of the specific values of global coding vectors. Intuitively, global mixing vectors can be regarded as a limited representation of global coding vectors. Given network mixing vectors, it may not be sufficient to tell whether a certain symbol can be decoded or not. Thus, using the network mixing representation, the extraneous flows, when mixed with the desired flows on the paths to each terminal, are not guaranteed to be cancelled at the terminal. For decodability to hold at all the terminals, the global mixing vectors at all edges must satisfy the following definition of feasibility for scalar linear network mixing.

**Definition 2 (Feasibility of Scalar Linear Network Mixing):** For a network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) and a set of flows \( \mathcal{P} \) with sources \( S \) and terminals \( T \), a linear network mixing design \( \{ x_{i,j} \in \{0,1\}^P : (i, j) \in \mathcal{E} \} \) is called feasible if the following three conditions are satisfied: 1) \( x_{s_p,j} = e_p \) for source edge \((s_p,j) \in \mathcal{E}\), where \( s_p \in S \) and \( p \in \mathbb{P} \); 2) \( x_{ij} = \bigvee_{k \in \mathcal{L}} \beta_{kij} x_{ki} \) for edge \((i,j) \in \mathcal{E}\) not outgoing from a source, where \( i \notin S \) and \( \beta_{kij} \in \{0,1\} \); 3) \( x_{i,t,p} = 0 \), where \( i \in \mathcal{I}_t \), \( p \notin \mathcal{P}_t \), \( t \in \mathcal{T} \).

Note that Condition 3) in Definition 2 ensures that for each terminal, the extraneous flows are not mixed with the desired flows on the paths to the terminal. In other words, linear mixing allows only mixing at intermediate nodes. This is not as general as using linear network coding, which allows mixing and canceling (i.e., removing one or multiple flows from a mixing of flows) at intermediate nodes.

Given a feasible linear network mixing design, one of the ways to implement mixing when \( \mathcal{F} \) is large is to use random linear network coding (RLNC) [3], as discussed in the introduction. Note that, in performing RLNC based on \( \beta_{kij} \), \( x_{i,j,p} \) can be randomly chosen in \( \mathcal{F} \) when \( \beta_{kij} = 1 \), but \( \alpha_{kij} \) has to be chosen to be 0 when \( \beta_{kij} = 0 \).

### III. MIXING PROBLEM FORMULATION

In this section, we formulate the problem of selecting mixing coefficients \( \{ \beta_{kij} \} \) and \( \{ x_{i,j,p} \} \) to minimize the cost of the subgraph involved in the coding solution, i.e., the set of edges used in delivering the flows.

**Problem 1 (Integer Flows with Mixing Only):**

\[
U^* \triangleq \min_{\{z_{ij}\},\{f_{i,j,p}\},\{x_{i,j,p}\},\{\beta_{kij}\}} \sum_{(i,j) \in \mathcal{E}} U_{ij}(z_{ij})
\]

s.t. \( z_{ij} \in \{0,1\} \) \( (i,j) \in \mathcal{E} \) \hspace{1cm} (3)
\( x_{ij,p} \in \{0,1\} \) \( (i,j) \in \mathcal{E}, p \in \mathcal{P} \) \hspace{1cm} (4)
\( \beta_{kij} \in \{0,1\} \) \( (k,i) \in \mathcal{E}, (i,j) \in \mathcal{E} \) \hspace{1cm} (5)
\( f_{i,j,p} \in \{0,1\} \) \( (i,j) \in \mathcal{E}, p \in \mathcal{P}_t, t \in \mathcal{T} \) \hspace{1cm} (6)
\( \sum_{p \in \mathcal{P}_t} f_{i,j,p} \leq z_{ij} \) \( (i,j) \in \mathcal{E}, t \in \mathcal{T} \) \hspace{1cm} (7)
\( \sum_{k \in \mathcal{C}_t} f_{ik,p} - \sum_{k \in \mathcal{C}_t} f_{ik,p}^t = \sigma_{i,p} \) \( i \in \mathcal{V}, p \in \mathcal{P}_t, t \in \mathcal{T} \) \hspace{1cm} (8)
\( f_{i,j,p} \leq x_{i,j,p} \) \( (i,j) \in \mathcal{E}, p \in \mathcal{P}_t, t \in \mathcal{T} \) \hspace{1cm} (9)
\( x_{s_p,j} = e_p, (s_p,j) \in \mathcal{E}, p \in \mathcal{P} \) \hspace{1cm} (10)
\( x_{ij} = \bigvee_{k \in \mathcal{L}} \beta_{kij} x_{ki} \) \( (i,j) \in \mathcal{E}, i \notin S \) \hspace{1cm} (11)
\( x_{i,t,p} = 0 \), \( i \in \mathcal{I}_t \), \( p \notin \mathcal{P}_t \), \( t \in \mathcal{T} \) \hspace{1cm} (12)

\( ^{6} \text{Note that } \bigvee \text{ denotes the “or” operator (logical disjunction).} \)
where  \( \sigma_{i,p} = \begin{cases} 1, & i = s_p \\ -1, & i = t \\ 0, & \text{otherwise} \end{cases}, \quad i \in V, \ p \in P_i, \ t \in T. \)

In the above formulation, \( z_{ij} \in \{0, 1\} \) indicates whether edge \((i, j) \in E\) is involved in the coding subgraph, and \( f_{ij,p} \in \{0, 1\} \) indicates whether edge \((i, j) \in E\) is involved in delivering flow \( p \in P_i \) to terminal \( t \in T \). For notational simplicity, we write \( \{z_{ij} : (i, j) \in E\}, \{f_{ij,p} : (i, j) \in E, p \in P_i \} \) and \( \{x_{ij,p} : (i, j) \in E, p \in P\} \) and \( \{k_{ij} : (i, j) \in E\} \) as \( \{z\}, \{f\} \) and \( \{x\} \) where there is no confusion.

**Remark 1 (Problem 1 for Multicast):** When \( P_1 = P \) for all \( t \in T \) (i.e., multicast), the constraint in (12) does not exist, and the constraint in (9) is always satisfied by choosing \( \{k_{ij} = 1\} \) and choosing \( \{x_{ij,p}\} \) accordingly by (10) and (11). Therefore, Problem 1 for general connections reduces to the conventional minimum-cost scalar time-invariant linear network code design problem for the multicast case.

![Fig. 1: Illustration of a feasible solution to Problem 1.](image)

Fig. 1 illustrates a feasible solution to Problem 1. \( P = \{1, 2\}, \ S = \{1, 2\}, \ T = \{7, 8, 10\}, \ P_1 = \{1\}, \ P_2 = P_3 = \{1, 2\}. \) Flow paths (sets of edges over which the flow variables are one) from the two sources (i.e., \( \{(i, j) \in E : f_{ij,p} = 1\} \) for all \( p \in P_i \) and \( t \in T \)) are illustrated using green and blue curves, respectively. Since \( P_2 = P_3 = \{1, 2\} \), the flows from \( s_1 \) to \( t_2 \) and \( s_2 \) to \( t_3 \) are allowed to be mixed at edge \((4, 6)\). The red edges carry network-coded information.

**Lemma 1:** Suppose Problem 1 is feasible. Then, for each feasible \( \{x_{ij,p}\} \) and \( \{k_{ij}\} \), there exists a feasible linear network code design \( \{\alpha_{kij}\} \) and \( \{c_{ij,p}\} \) with a field size \( F > T \) to deliver the desired flows to each terminal.

**Proof:** Please refer to Appendix A.\(^7\)

\(^7\)The proof is based on the mixing coefficients we introduced and their relationship with network coding coefficients. If \( k_{ij} = 0 \) (or \( x_{ij,p} = 0 \)), then \( \alpha_{kij} = 0 \) (or \( c_{ij,p} = 0 \)).

Next, the minimum network cost of Problem 1 is no greater than the minimum costs of the two-step mixing approach for general connections in [4] and routing for integer flows, due to the following reasons. Problem 1 with an extra constraint \( \sum_{p \in P} x_{ij,p} \in \{0, 1\} \) for all \( (i, j) \in E \) is equivalent to the minimum-cost routing problem. Problem 1 with \( \beta_{kij} \) for all \( (k, i), (i, j) \in E \), instead of (5), is equivalent to the minimum-cost flow rate control problem in the second step of the two-step mixing approach for general connections in [4].

Finally, we discuss the complexity of Problem 1. The cardinality of \( \{k_{ij}\} \) is \( \sum_{i \in V} O_j = \sum_{j \in V} I_j O_j \leq \sum_{j \in V} DO_j = DE \). The cardinality of \( \{f_{ij,p}\} \) is smaller than or equal to \( PTE \). The cardinalities of \( \{z_{ij}\} \) and \( \{x_{ij,p}\} \) are \( E \) and \( PE \), respectively. Therefore, the total number of variables in Problem 1 is smaller than or equal to \( (D + 1)E + (T + 1)PE \), i.e., polynomial in \( E, T \), and \( P \). Problem 1 is a binary optimization problem and is NP-complete in general.

**IV. A Probabilistic Distributed Algorithm**

Obtaining a feasible solution to Problem 1 can be directly treated as a CSP [10] (see Appendix B for a background on CSPs). Specifically, \( \{z_{ij}\} \cup \{f_{ij,p}\} \cup \{x_{ij,p}\} \cup \{\beta_{kij}\} \) and \( \{0, 1\} \) can be treated as the variables and the finite set of the CSP. Constraints (7)-(12) can be treated as the clauses of the CSP. While CSPs are in general NP-complete, several centralized CSP solvers (see references in [10]) and the distributed CSP solver proposed in [10] can be applied to solve this (naive) CSP. However, the direct application of the distributed CSP solver in [10] leads to high complexity due to the large constraint set. In this section, by exploiting the features of the constraints in Problem 1, we obtain a different CSP and present a probabilistic distributed solution with a significantly reduced number of variables.

First, we construct a new problem, which we show to be a CSP. This new problem is better suited than the original problem to being treated using a probabilistic distributed algorithm based on the distributed CSP solver presented in [10]. Combining (3) and (7), we have an equivalent constraint purely in terms of \( \{f_{ij,p}\} \), i.e.,

\[
\sum_{p \in P_i} f_{ij,p} \in \{0, 1\}, \quad (i, j) \in E, \ t \in T. \quad (13)
\]

In addition, from (11), we have an equivalent constraint purely in terms of \( \{x_{ij,p}\} \), i.e.,

\[
\exists \beta_{kij} \in \{0, 1\} \forall k \in I, \text{ s.t. } x_{ij,p} = \max_{k \in I} \beta_{kij} x_{ki}, \quad (i, j) \in E, \ t \in T. \quad (14)
\]

Therefore, we can solve only for the variables \( \{f_{ij,p}\} \cup \{x_{ij,p}\} \) in a distributed way, as \( z_{ij} \) can be obtained directly from a feasible set \( \{f_{ij,p}\} \) by choosing \( z_{ij} = \max_{k \in I} \beta_{kij} x_{ki} \) according to (3) and (7), and \( \{\beta_{kij}\} \) can be obtained from a feasible set \( \{x_{ij,p}\} \) by (10) and (11). We group all the local
variables for each edge \((i, j) \in \mathcal{E}\) and introduce the vector variable \((\bar{f}_{ij}, x_{ij}) \in \Lambda_{ij}\), where \(f_{ij} \triangleq (f^i_{ij})_{t \in \mathcal{T}}\), \(f^i_{ij} \triangleq (f_{ij}(t))_{p \in \mathcal{P}_t}\), and \(\Lambda_{ij} \triangleq \{(\bar{f}_{ij}, x_{ij}) : (4), (6), (9), (10), (12), (13)\}. We now consider a new CSP, different from the naïve one that would be obtained from the original optimization problem. We treat \((\bar{f}_{ij}, x_{ij})\) and \(\Lambda_{ij}\) as the variable and the finite set for edge \((i, j)\) of the CSP. We write the clauses for \(\{\bar{f}_{ij}, x_{ij}\}\) as follows:

\[
\phi^f_i (f_i) = \begin{cases} 1, & \text{if } (8) \text{ holds } \forall p \in \mathcal{P}_t, \ t \in \mathcal{T}, \ i \in \mathcal{V} \\ 0, & \text{otherwise} \end{cases} \tag{15}
\]

\[
\phi^{x}_{ij} (x_{ij}, \{x_{ki} : k \in \mathcal{I}_i\}) = \begin{cases} 1, & \text{if } (14) \text{ holds } \\ 0, & \text{otherwise} \end{cases}, \quad (i, j) \in \mathcal{E}, i \notin \mathcal{S} \tag{16}
\]

where \(f_i \triangleq \{f_{ik} : k \in \mathcal{O}_i\} \cup \{f_{ki} : k \in \mathcal{I}_i\}\). It can be seen that the local constraints in (4), (6), (9), (10), (12) and (13) (i.e., (3) and (7)) are considered in the finite set \(\Lambda_{ij}\) of the CSP with respect to each edge \((i, j) \in \mathcal{E}\). On the other hand, the non-local constraints in (8) and (14) are considered in clauses \(\phi^f_i\) in (15) and \(\phi^{x}_{ij}\) in (16), respectively. Therefore, the CSP has considered all the constraints in Problem 1. Note that the number of variables \((E)\) and the number of clauses \((\leq V + E - P)\) of the new CSP are much smaller than the number of variables \((\leq (1 + D + P + T)E)\) and the number of clauses \((\leq (1 + T + T)E + TPV + TPD)\) of the naïve CSP mentioned above. This feature will favor the complexity reduction of a distributed solution based on the distributed CSP solver in [10].

Next, we construct the clause partition. The set of clauses in which variable \((\bar{f}_{ij}, x_{ij})\) participates is

\[
\Phi_{ij} = \left\{\phi^f_i, \phi^f_j\right\} \cup \left\{\phi^{x}_{ij}, \phi^{x}_{jk} : i \notin \mathcal{S}, k \in \mathcal{O}_j\right\}, \quad (i, j) \in \mathcal{E}. \tag{17}
\]

Now, the new CSP can be solved using the distributed iterative CFL algorithm [10, Algorithm 1] based on the clause partition. Specifically, each edge \((i, j) \in \mathcal{E}\) realizes a random variable selecting \((f_{ij}, x_{ij})\). Allow message passing on \((f_{ij}, x_{ij})\) between adjacent nodes to evaluate the related clauses. Based on whether the clauses in (17) are satisfied or not, the distribution of the random variable of each edge \((i, j) \in \mathcal{E}\) is updated.

Algorithm 1 represents a distributed algorithm to solve Problem 1 from feasible solutions obtained to the related CSP using CFL. Based on the convergence result of CFL [10, Corollary 2], we can easily see that \(U_l \mapsto U^*\) almost surely as \(l \to \infty\), if Problem 1 is feasible. The convergence performance of Algorithm 1 is illustrated in Appendix C. In the future work, we shall focus on improving the convergence performance of Algorithm 1 by improving CFL.

In Step 3, CFL is run for a sufficiently long time. Step 4 (Step 6) can be implemented with a master node obtaining the network convergence information of CFL (network cost) from all nodes or with all nodes computing the average convergence indicator of CFL (average network cost) locally via a gossip algorithm.

V. Conclusion

In this paper, we introduce linear network mixing coefficients for code constructions of general connections. For such code constructions, we pose the problem of cost minimization for the subgraph involved in the coding solution, and relate this minimization to a CSP. We present a probabilistic distributed algorithm with almost sure convergence in finite time by applying CFL. Our approach allows fairly general coding across flows, guarantees no greater cost than routing, and demonstrates a possible distributed implementation.

References

APPENDIX A: PROOF OF LEMMA 1

Let \( \{z_{ij}\}, \{x_{ij,p}\}, \{\beta_{kij}\} \) and \( \{f_{ij}^t\} \) denote a feasible solution to Problem 1. Note that \( \{x_{ij,p}\} \) is uniquely determined by \( \{\beta_{kij}\} \) according to (10) and (11), which correspond to Conditions 1) and 2) in Definition 2. In addition, by (12), which corresponds to Condition 3) in Definition 2, we know that \( \{x_{ij,p}\} \) ensures that for each terminal, the extraneous flows are not mixed with the desired flows on the paths to the terminal. We shall show that based on \( \{\beta_{kij}\} \), we can find local coding coefficients \( \{\alpha_{kij}\} \), which uniquely determine feasible global coding coefficients \( \{c_{ij,p}\} \) according to

\[
\begin{align*}
    c_{sp,j} &= e_p, \quad (s_p,j) \in \mathcal{E}, \quad p \in \mathcal{P} \\
    c_{ij} &= \sum_{k \in \mathcal{I}_i} \alpha_{kij} c_{k,i}, \quad (i,j) \in \mathcal{E}, i \notin \mathcal{S}.
\end{align*}
\]

(18)

(19)

Note that (18) and (19) correspond to Conditions 1) and 2) in Definition 1).

First, we choose \( \alpha_{kij} = 0 \) if \( \beta_{kij} = 0 \). Note that as a feasible solution, \( \{x_{ij,p}\} \) is uniquely determined by \( \{\beta_{kij}\} \) according to (10) and (11). In addition, we choose \( \{c_{ij,p}\} \) based on \( \{\alpha_{kij}\} \) according to (18) and (19). Thus, by (18), (19), (10) and (11), we can show that \( c_{ij,p} = 0 \) if \( x_{ij,p} \) is by induction. Thus, by (12), we have

\[
c_{it,p} = 0, \quad i \in \mathcal{I}_t, \quad p \notin \mathcal{P}_t, \quad t \in \mathcal{T}.
\]

(20)

In other words, each terminal \( t \in \mathcal{T} \) only needs to consider \( \{c_{it,p} : i \in \mathcal{I}_t, p \in \mathcal{P}_t\} \) for decoding. By (8), we can form a flow path from source \( s_p \) to terminal \( t \), which consists of the edges in \( \mathcal{E}_p = \{(i,j) \in \mathcal{E} : f^t_{ij,p} = 1\} \), where \( p \in \mathcal{P}_t \). By (3) and (7), we know that for all \( t \in \mathcal{T} \), there exists \( P_t \) edge-disjoint unit flow paths, each one from one source \( s_p \) to terminal \( t \), where \( p \in \mathcal{P}_t \). Note that \( \{x_{ij,p}\} \) satisfies all the conditions in Definition 2. Thus, by (9), we know that all the flow paths satisfy that for each terminal, the extraneous flows (information) are not mixed with the desired flows (information) on the flow paths to the terminal. Let \( A_t \) denote the \( P_t \times P_t \) matrix, each row (out of \( P_t \) rows) of which consists of the \( P_t \) elements in \( \{c_{it,p} : p \in \mathcal{P}_t\} \) for the last edge \( (i,t) \) on one flow path (out of \( P_t \) flow paths) to terminal \( t \), where \( i \in \mathcal{I}_t \). Note that \( A_t \) (in terms of \( \{c_{it,p} : i \in \mathcal{I}_t, p \in \mathcal{P}_t\} \) for all \( P_t \) flow paths) can also be expressed in terms of local coding coefficients \( \{\alpha_{kij}\} \) by (18) and (19). By (18) and (19), we know that 1) and 2) of Definition 1 are satisfied. Therefore, it remains to show that 3) of Definition 1 is satisfied. This can be achieved by choosing \( \{\alpha_{kij} \in \mathcal{F} : \beta_{kij} \neq 0\} \) so that \( A_t \) for all \( t \in \mathcal{T} \) are full rank, which is equivalent to the requirement that \( y(\{\alpha_{kij} : \beta_{kij} \neq 0\}) \neq 0 \) \[11, Page 19-20\], where

\[
y(\{\alpha_{kij} : \beta_{kij} \neq 0\}) \triangleq \prod_{t \in \mathcal{T}} \det(A_t).
\]

(21)

Given all the local coding coefficients \( \{\alpha_{kij}\} \), we can compute global coding coefficients \( \{c_{ij,p}\} \), and vice versa.

APPENDIX B: BACKGROUND ON DECENTRALIZED CONSTRAINT SATISFACTION PROBLEM

In Section IV, we have shown that Problem 1 is related to a CSP and can be solved in a distributed manner using recent CFL for CSP if we allow message passing regarding flow rates and global mixing coefficients between adjacent nodes. In this part, we review some existing results on CSP in [10].

First, we define CSP as below [10].

Definition 3 (Constraint Satisfaction Problem): A CSP consists of \( N \) variables \( \{\lambda_1, \cdots, \lambda_N\} \) and \( C \) clauses \( \{\phi_1, \cdots, \phi_C\} \). Each variable \( \lambda_n \) takes values in a finite set \( \Lambda \), i.e., \( \lambda_n \in \Lambda \) for all \( n \in N \). Let \( \Lambda \triangleq (\lambda_1, \cdots, \lambda_N) \in \Lambda^N \). Each clause \( c \in C \) is a function \( \phi_c : \Lambda^N \rightarrow \{0, 1\} \), where for an assignment of variables \( \lambda \in \Lambda^N \), \( \phi_c(\lambda) = 1 \) if clause \( c \) is satisfied and \( \phi_c(\lambda) = 0 \) otherwise. An assignment \( \lambda \in \Lambda^N \) is a solution to the CSP if and only if all clauses are simultaneously satisfied, i.e.,

\[
\min_{c \in C} \phi_c(\lambda) = 1.
\]

(22)

Next, we show that if \( F > T \), we can choose \( \{\alpha_{kij} \in \mathcal{F} : \beta_{kij} \neq 0\} \) such that \( y(\{\alpha_{kij} : \beta_{kij} \neq 0\}) \neq 0 \). We first show that for all \( t \in \mathcal{T} \), \( \det(A_t) \) is not identically equal to zero. For all \( p \in \mathcal{P}_t \) and \( t \in \mathcal{T} \), choose \( \alpha_{kij} = 1 \) for all edges \( (k,i), (i,j) \in \mathcal{E} \) on the flow path from source \( s_p \) to terminal \( t \), i.e., \( (k,i),(i,j) \in \mathcal{E}_p \), and \( \beta_{kij} = 0 \) for all edges \( (k,i), (i,j) \in \mathcal{E} \) not on the same flow path, i.e., \( \beta_{kij} \neq 0 \) if clauses \( \beta_{kij} \notin \mathcal{E}_p \). This local coding coefficient assignment makes \( A_t \) a \( P_t \times P_t \) identity matrix. Thus, \( \det(A_t) \) is not identically equal to zero [11, Page 20]. Then, we show that \( y(\{\alpha_{kij} : \beta_{kij} \neq 0\}) \) is not identically equal to zero. Similarly to [11, Page 31-32], we can write \( A_t = C_t(I - A)^{-1}B_t \). Here, \( A \) is an \( E \times E \) matrix common for all terminals and reflects the way the memory elements are connected. An element of \( A \) is either zero or an unknown variable in \( \{\alpha_{kij} : \beta_{kij} \neq 0\} \). \( C_t \) is a \( P_t \times E \) matrix expressing the outputs terminal \( t \) observes depending on the state variables. \( B_t \) is an \( E \times P_t \) matrix expressing the outputs terminal \( t \) observes depending on the inputs.\(^{10}\)

\(^{10}\)Note that \( C_t \) can be treated as the \( P_t \) rows of \( C_j \) in [11, Page 31-32] corresponding to the last edges of the \( P_t \) edge-disjoint flow paths to terminal \( t \). \( B_t \) can be treated as the \( P_t \) columns of \( B \) in [11, Page 31-32] corresponding to the \( P_t \) sources for terminal \( t \).
To solve a CSP in a distributed way, clause participation is introduced [10]. Let \( \Lambda \) be the set of clauses involved in a CSP. For each variable \( \lambda_n \), let \( C_n \) denote the set of clause indices in which it participates, i.e., \( C_n = \bigcup_{n=1}^{\Lambda} \{ i : \min_{\lambda_n \in \Lambda} \phi(i, \lambda_n) = 0, \max_{\lambda_n \in \Lambda} \phi(i, \lambda_n) = 1 \} \). Thus, we can rewrite the left hand side of (22) in a way that focuses on the satisfaction of each variable, i.e.,

\[
\min_{n \in \mathcal{N}} \min_{c \in C_n} \phi_c(\lambda) = 1. \tag{23}
\]

The form in (23) enables us to solve CSPs in a distributed iterative way by locally evaluating the clauses in \( C_n \) and then updating \( \lambda_n \). The Communication-Free Learning (CFL) algorithm from [10] is a distributed iterative algorithm which can find a satisfying assignment to a CSP almost surely in finite time.

Note that most CSPs are NP-hard. The best known CSP solvers are designed for centralized problems. The key contribution of CFL is enabling a distributed solution (see [10] for a detailed discussion).

APPENDIX C: CONVERGENCE OF ALGORITHM 1

We illustrate the almost sure convergence of the proposed probabilistic distributed Algorithm 1 using the following example. Consider multicast in the butterfly network, as illustrated in Fig. 2. Fig. 3 illustrates the convergence performance of the CFL in Algorithm 1 over edge (3,4) for one run. We can see that the CFL in Algorithm 1 converges (when all the probabilities go to 1 or 0) [10]. Here, the CFL converges to \( x_{34,1} = 1, x_{34,2} = 1, f_{34,1}^1 = 0, f_{34,1}^2 = 1, f_{34,2}^1 = 1, f_{34,2}^2 = 0 \). Note that edge (3,4) is the coding edge for multicast in the butterfly network. Therefore, Algorithm 1 converges to a feasible solution, i.e., the optimal solution in this case.

Fig. 2: Multicast in the butterfly network. The feasible \( \{x_{ij,p}\} \) and \( \{\beta_{kj}\} \) are shown in the figure. The feasible \( \{f_{ij,p}^1\} \) and \( \{z_{ij}\} \) can be seen from the flow paths from the two sources to the two terminals, illustrated using green and blue curves, respectively.

Fig. 3: Convergence performance of the CFL in Algorithm 1 over edge (3,4). Correct Event denotes \( x_{34,1} = 1, x_{34,2} = 1, f_{34,1}^1 = 0, f_{34,1}^2 = 1, f_{34,2}^1 = 1, f_{34,2}^2 = 0 \), and corresponds to a feasible solution. Incorrect Event 1 denotes \( x_{34,1} = 1, x_{34,2} = 1, f_{34,1}^1 = 0, f_{34,1}^2 = 1, f_{34,2}^1 = 1, f_{34,2}^2 = 1 \). Incorrect Event 2 denotes \( x_{34,1} = 1, x_{34,2} = 1, f_{34,1}^1 = 0, f_{34,1}^2 = 0, f_{34,2}^1 = 1, f_{34,2}^2 = 1 \). Note that there are 18 possible events, i.e., \( |\Lambda_{34}| = 18 \).