Ultrahigh Precision Absolute and Relative Rotation Sensing using Slow and Fast Light

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In this paper, we show a passive resonator based optical gyroscope whose sensitivity can be enhanced by many orders of magnitude by using a characteristic anomalous dispersion that corresponds to superluminal light propagation in the medium. Dispersion bounds for sensitivity enhancement are quantified by defining the critical anomalous dispersion (CAD) condition that can be experimentally realized using bi-frequency pumped Raman gain in an atomic medium. In contrary, we also determined ultra slow light based dispersion for relative rotation sensing in a specially designed configuration of Sagnac interferometer that utilizes dispersive drag phenomena in presence of Doppler shift. Rotational sensitivity of these interferometers scales as ratio of phase velocity to group velocity, which could be many orders magnitude higher than a conventional optical gyroscope.

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Recent experiments have demonstrated significantly large "slow" and "superluminal" group velocity of light compared to the free-space velocity in dispersive material medium (e.g. dilute vapor or solid), using processes such as electromagnetically induced transparency (EIT), photorefractive wave-mixing, bi-frequency Raman gain and coherent population oscillation etc [1-5]. Extreme dispersion in "slow light" medium has been recently considered to boost the rotational sensitivity of optical gyro via Sagnac effect [6]. However, the mechanism by which strong material dispersion affects relativistic Sagnac effect, has remained largely elusive. It is commonly perceived that the Sagnac time delay or phase shift in presence of a dielectric medium is independent of the refractive property of the medium [7,8]. Although this is true for a co-moving medium and absolute inertial rotation, it is not true if in particular the medium is moving or accelerating with respect to the inertial frame of rotation. In a short communication, Leonhardt and Piwnicki [6] predicted the idea of slow light interferometer gyroscope, without explicitly validating their results for specific cases where the medium can be in relative motion. Later, Zimmer and Fleischhauer [9] refuted the idea of slow light based sensitivity enhancement and proposed an alternative hybrid light and matter-wave interferometer, based on slow light momentum transfer from light to atoms.

In an interferometric optical gyroscope that allows relative motion between the medium and interferometer, dispersive drag coefficient can influence the Sagnac fringe shift. Precise experimental measurement of light drag in moving optical medium has been of considerable interest to scientific community as it has a profound influence on our fundamental perception of space and time. On the other hand, medium correction due to light drag on optical Sagnac effect has been studied and interpreted [8,10] long before the inception of resonant dispersive “slow light” phenomena. Here, we use axioms of special theory of relativity in the rotating frame of reference of a Sagnac interferometer to describe possible light dragging phenomena in a moving dielectric medium. In this case, we also show that use of slow light can hugely enhance the rotational sensitivity by a factor as large as slowing in the medium. Such a system is most appropriate for precise sensing of relative rotation of a local system. We also consider a passive resonator based gyroscope for absolute rotation sensing where the Sagnac effect manifests in terms of frequency splitting of cavity modes. Unlike an interferometer, frequency-splitting in a resonator gyroscope is governed by optical path length that is dependent on the refractive property of the medium even in a Doppler free situation [10-12]. Using a self-consistent analysis based on special relativity (SR), we also discuss the effect of extreme dispersion on a passive resonator gyro and present an important observable effect that suggests many orders of magnitude increase in rotation sensitivity using critically anomalous dispersion (CAD) under superluminal light propagation in the medium. Such a system can be experimentally implemented using controllable dispersion bounds induced by bi-frequency pumped Raman gain in atomic medium [13], and can be used in navigational or geophysical applications for ultrahigh precision absolute inertial rotation sensing.

Experimental predictions of Sagnac effect can be described within the theoretical framework of special relativity, without necessarily invoking general relativity (GR). Application of GR is manifestly necessary only in presence of significant gradient in the gravitational potential that introduces space-time curvature [8]. However, this is not the case for Sagnac effect curvature. In what follows, we briefly discuss how the SR encompasses Sagnac effect. SR velocity addition rule is used to describe propagation of phase fronts (PFs) in the rotating frame and subsequently, to calculate the time delay and phase shift in the inertial
frame of rotation. We extend the first-order special relativistic proof in a consistent manner to describe the effect of dispersive and moving media on rotation sensing using light dragging phenomena.

It is known that optical, acoustic, or matter-wave interferometry can be employed to measure absolute inertial rotation around an axis perpendicular to the plane of an interferometer, via the Sagnac effect. For simplicity, let us assume waves in a Mach-Zehnder interferometer (MZI) shown in Fig. 1, that are constrained to propagate in a circular path with a radius \( R \). In general, the wave could be of any kind e.g. optical, matter or acoustic wave. Using SR in an inertial frame for the clockwise \( CW(+) \) and counter-clockwise \( CCW(-) \) directions, the relativistic velocities \( V^\pm_R \) of the PFs, the time \( T^\pm \) of PFs to travel from BS1 to BS2 and the effective distances \( L^\pm \) from the BS1 to BS2 are related as follows

\[
V^\pm_R = \frac{V_p \pm v}{1 \pm \frac{v}{c} \frac{v}{C_o}} , \quad L^\pm = \pi R \pm v T^\pm , \quad T^\pm = \frac{L^\pm}{V^\pm_R} \quad (1)
\]

where \( V_p \) is the velocity of PF in absence of rotation, \( C_o \) is the velocity of light in vacuum, \( v = \Omega R \) is the tangential velocity of rotation, and \( \Omega \) is the rotation rate.

![Figure 1](https://example.com/figure1.png)

**Figure 1** Schematic illustration of Mach-Zehnder type Sagnac interferometer with counter-propagating beams in a circular optical path

Thus, the time delay between the propagating waves and their relative phase shift at the detector are given by

\[
\Delta t_o = T^+ - T^- = \pi R \left[ \frac{V_R^+ - V_R^-}{V_R^+ + v} \right] + 2 \frac{v}{\sqrt{1 - \beta^2}} \sqrt{V_R^+ - V_R^-} \approx \frac{2 \lambda R}{C_o}, \quad \beta = \frac{v}{C_o} \ll 1
\]

\[
\Delta \phi_o = \omega \Delta t_o = \frac{4 \pi \lambda R}{C_o} \quad (2)
\]

where \( \lambda = \pi R^2 \) is the enclosed area, \( \omega \) the angular frequency, \( \lambda_o \) the vacuum wavelength of the wave and \( \beta \) is the boost parameter, assumed to be much less than unity for typical rotations. The form of time delay in eqn. 2 due to Sagnac effect attests to the universal fact that the time delay is simply a geometric effect, attributable to relativistic time dilations and does not depend at all on the velocity of the wave. It only involves the free space velocity of light, even if acoustic waves or matter waves are used. However, matter wave interferometry had fascinating implications on Sagnac rotation sensing, as the effect for a given rotation rate \( \Omega \) is enhanced due to small wavelength and low particle speed of the matter wave. Familiar result of matter-wave Sagnac phase shift \( \Delta \phi = 4 \pi m \lambda \Omega \omega / h \), where \( m \) is the rest mass, can be derived from eqn. 2 by substituting matter wave wavelength by the so-called Compton wavelength.

In presence of optical waves in Sagnac interferometer consisting of a refractive medium, the generic phase shift \( \Delta \phi \), in eqn. 2 is independent of the refractive index of the medium. This result is somewhat surprising, since it is expected that besides rotation induced delay, the light also experiences additional drag in moving dielectric medium. Thus the travel time of light around the optical circuit should change and depend on medium index \( n \). However, this is not true for a co-moving or co-rotating medium in Sagnac interferometer. One can explicitly show this by considering relativistic velocities \( V^\pm_R \) of the CW and CCW phase fronts (eqn. 1), in the limit of \( \beta \ll 1 \)

\[
V^\pm_R = \frac{C_o}{n} \pm v \alpha_p, \quad \alpha_p = \left(1 - \frac{1}{n^2}\right) \quad (3)
\]

where \( C_o / n \) is now the velocity \( V_p \) of phase fronts in absence of rotation, and the term \( \alpha_p \) is known as the Fresnel drag coefficient. Using eqns. 2, one can write the time delay and the phase shift in a generalized form

\[
\Delta t = n^2 \left(1 - \alpha_p\right) \Delta t_o, \quad \Delta \phi = n^2 \left(1 - \alpha_p\right) \Delta \phi_o \quad (4)
\]

Substituting \( \alpha_p \) from eqn. 3 in eqn. 4, one gets the same results as in eqn. 2. This is because the previous derivation using SR already includes the effect of moving medium on light propagation and we simply re-expressed the results in a different way. One also concludes that SR incorporates drag induced medium correction in Sagnac effect. However, its magnitude is independent of Fresnel drag coefficient, as well as the refractive index for a Sagnac interferometer that uses a common frame of rotation for light source, interferometer as well as the medium. Such a system is commonly employed for absolute rotation sensing in navigational systems. This system does not possess Doppler of shift of any kind between the medium and the source or interferometer, since they are all co-rotating at the same rate \( \Omega \).
One can relax this constraint and allow several modifications of Sagnac interferometer where relative motion between the frame and the medium introduces a Doppler shift. As we shall see, under this condition, light drag resulting from dispersion affects the final fringe shift. Such a system, however, can only be used in relative rotation sensing. Fig. 2A shows a similar MZI considered for Sagnac interferometry, that allows relative motion of medium with respect to the interferometer. In order to make the system simpler, the connection with the first BS is established using a flexible fiber, thus making the MZI rotate without rotating the source. Although, stationary source in presence of co-rotating frame and medium introduces Doppler shift at the first BS, consequently, this shift does not have a first order effect on the phase shift since the BS effectively serves as the source for both arms. Therefore, the fringe shift described in eqn. 2 still holds. We assume that the medium inside the MZI paths is allowed to move at a velocity $V_M$ with respect to the MZI frame (i.e., the mirrors and beam splitters). As seen by the medium, the CW(+) and CCW(-) beams are now Doppler shifted by equal and opposite amounts, given by $\Delta \omega^2 = \pm \omega V_M / C_o$. Expressions for relativistic velocities can be given by including first-order dispersion in $n(\omega)$ as

$$V^\pm_k = \frac{C_o}{n_o} \left( 1 - \frac{\Delta \omega^2}{n_o} \right) \mp \nu \alpha_F = \frac{C_o}{n_o} \mp \frac{V_M}{n_o^2} \frac{\partial n}{\partial \omega} \mp \nu \alpha_F$$

$$= \frac{C_o}{n_o} \mp \frac{V_M}{n_o^2} \left( \frac{n_2 - n_o}{n_2} \right) \mp \nu \alpha_F$$

(5)

where $n_o$ is the phase index at the input frequency of $\omega$, $\alpha_F = (1 - 1/n_2)$ is the Fresnel drag coefficient, and $n_2 = \langle n_o + \omega \partial n / \partial \omega \rangle$ is the group index, defined as the ratio of the free space velocity of light to the group velocity at $\omega$. One can create experimental situations where the medium is stationary or non-comoving, as illustrated in Fig. 3B, and 3C, giving rise to $V_M = (-\nu)$. The relativistic velocities in this case are given by

$$V^\pm_k = \frac{C_o}{n_o} \pm \nu \alpha_L, \quad \alpha_L = \left[ 1 - \frac{1}{n_o^2} + \frac{(n_2 - n_o)}{n_o^2} \right]$$

(6)

where $\alpha_L$ is the so-called Laub drag coefficient, which has been experimentally measured with great precision using a passive ring resonator technique[14]. The expressions for time delay and phase shift are obtained as

$$\Delta t = \left( n_2 \left( 1 - \alpha_L \right) \right) \Delta t_o, \quad \Delta \phi = \left( n_2 \left( 1 - \alpha_L \right) \right) \Delta \phi_o$$

(7)

Unlike the Doppler free situation, the coefficients of time delay and phase shift depend on the refractive property of the medium. The result is consistent as in the absence of dispersion (i.e., $n_o = n_0$), one recovers the same results as in eqn. 2. In presence of a highly dispersive medium with $n_2 > n_o$ (i.e. $\partial n / \partial \omega >> n_o / \omega$), a characteristic of extreme normal dispersion in the medium that produces “slow-light”, the time delay and phase shift are given by

$$\Delta t = n_2 \Delta t_o, \quad \Delta \phi = n_2 \Delta \phi_o$$

(8)

Thus, the rotation sensitivity of a slow light based gyroscope scales directly with the group index $n_2$. Experimental observation of slow light reported in dressed atomic and solid media using EIT [1,2] have shown group index $n_2$ to be as large as $10^8$. Thus one can conceivably design an ultra-precise rotation sensor for sensing relative rotation of a local system using a uniformly spinning slow light medium in the interferometric beam path. A common path interferometer design can be used to null undesirable geometrical path induced errors that may result due to the motion of the medium.
be employed in sensing absolute rotation. This led us to consider the Sagnac effect in a passive ring cavity or resonator. Unlike the interferometer where the frequency of light remains unchanged by rotation, the frequencies of the cavity modes in a resonator get modified as a result of rotation. Figure 3 shows the schematic of a passive ring cavity (PRC) along with the necessary detection mode and servo mechanism to be used as a gyroscope. For simplicity, we assume the light path in the resonator to be circular with a radius \( R \) and same enclosed area \( A \). In absence of rotation, the light in the cavity resonates in both directions with a frequency \( \omega_o \), given by

\[
\omega_o = \frac{C_o}{n_o} \frac{2\pi N}{P}
\]

where \( N \) is an integer, and \( P = 2\pi R \) is the perimeter. In presence of rotation, the resonance frequencies (adjacent to \( \omega_o \)) are different for the CW(+) and CCW(-) directions, and are given by

\[
\omega^\pm = V^\pm \cdot \frac{2\pi N}{P}; \quad V^\pm = V_R^\pm + \nu
\]

\[
\Delta \omega_o = (\omega^+ - \omega^-) = \frac{\omega_o}{C_o n_o} (2\Omega R) = \frac{\omega_o}{C_o n_o} \cdot 4\Omega A \]

where \( V_R^\pm \) are the relativistic velocities given in eqn. 3, \( V^\pm_E \) are the effective velocities and \( A = \pi R^2 \) is the area enclosed by the resonator. Frequency difference \( \Delta \omega_o \), which is proportional to the rotation rate \( \Omega \), can be experimentally measured from heterodyne beat note of transmitted signals by adjusting the frequency of AOM1(AOM2) to maximize the cavity transmission in CCW(CW) direction.

It is important to note that even though the basic mechanisms for the MZI and the PRC based gyroscope are the same, the Sagnac frequency splitting of cavity mode explicitly depends on the refractive index of the medium. The expression for the beat frequency in eqn. 10 is strictly derived assuming phase index to be independent of frequency i.e. when the medium exhibits no dispersion. When the effect of dispersion is taken into account, the result changes significantly. Without any loss of generality, one can write

\[
\omega^\pm = \omega_o \pm \frac{\Delta \omega}{2} = V^\pm_E \frac{2\pi N}{P}
\]

where \( \Delta \omega \) is a parameter whose magnitude is to be determined. The effective velocities, \( V^\pm_E \) can be written as

\[
V^\pm_E = V_R^\pm + \nu = \frac{C_o}{n(\omega^\pm)} \left[ 1 \pm \frac{\nu}{C_o n(\omega^\pm)} \right]
\]

Expanding the value of \( n(\omega) \) around \( n_o \),

\[
V^\pm_E = \frac{C_o}{n_o} \left[ 1 \pm \frac{\nu}{C_o n_o} \right]^n \frac{\Delta \omega}{2}
\]

Substituting eqn. 13 in eqn. 11, one gets a self-consistent expression involving \( \Delta \omega \) that yields

\[
\Delta \omega = \frac{\Delta \omega_o}{1 + \omega_o n_o} = \frac{\Delta \omega_o}{n_o n_o} \frac{n_o}{n_o}
\]

For a medium that exhibits slow-light, \( n_o > n_o \) [i.e., \( \partial n / \partial \omega \gg (n_o / \omega_o) \)], where \( n_o \) is near unity resonant index of the medium], result in eqn. 14 implies a reduction in rotational sensitivity. On the other hand, it is equally possible to achieve a condition where \( 0 < n_o \ll 1 \), that corresponds to a characteristic anomalous dispersion \( (\partial n / \partial \omega < 0) \) leading to “superluminal” light propagation in the medium. Experiments are done recently using large spectrally narrow negative dispersion slope to demonstrate superluminal light propagation that does not necessarily violate the causality [4].

**Figure 4** Variation of sensitivity enhancement factor in the vicinity of CAD condition

Experimentally, if one chooses to operate near the condition \( 0 < n_o \ll 1 \), significant enhancement of sensitivity can be achieved. We define an enhancement factor \( \xi = n_o / n_o \) in eqn. 14 to quantify the bounds of this enhancement. To realize large enhancement in sensitivity, one has to experimentally operate close to the limit when the group index \( n_g \) of the medium approaches zero [i.e. \( \partial n / \partial \omega \approx (n_o / \omega_o) \)], the condition which we term as critically anomalous dispersion (CAD). However, the factor \( \xi \) becomes unphysical when the group index \( n_g \) approaches zero. To justify it, we extend our previous analysis to include the next higher order dispersion over the bandwidth \( \Delta \omega \) and the frequency shift is given by
\[ \Delta \omega = \Delta \omega_0 \left( 1 \pm \frac{2 \xi}{1 \pm \sqrt{1 + Q \xi^2}} \right), \quad Q = f \frac{n'' \omega^2}{\omega^2} = \left[ \frac{\partial^2 n}{\partial \omega^2} \right] / n_o \]

(15)

where \( f = \frac{\Delta \omega}{\omega_0} \) is a dimensionless quantity defined as the fractional bandwidth. In the limit when \( \xi \to \infty \) or \( \left( n_g \to 0 \right) \), non-zero value of \( Q \) (assuming \( |Q| = 1 \)) prevents the enhancement factor near the CAD condition to become asymptotic. Figure 4 shows a plot for sensitivity enhancement over a restricted range of \( \left( \frac{1}{\xi} = n_g / n_o \right) \) values choosing appropriate signs from eqn. (15) for positive and negative values of \( n_g \). The enhancement bound saturates to a maximum value \( 2/\sqrt{Q} \). This is also true in an experimental situation where the value of saturation is governed by inverse magnitude of second order dispersion over the dispersive bandwidth. As mentioned earlier, to realize large enhancement in sensitivity, one has to experimentally operate close to the vicinity of CAD condition. Linear growth of enhancement with \( \xi \) can be further explicitly quantified in the limit \( \xi^2 = \frac{1}{Q} \) and is given by

\[ \Delta \omega = \Delta \omega_0 \cdot \xi, \quad \frac{\partial n}{\partial \omega} = -\left( n_g / \omega_0 \right) \left[ 1 - \xi^{-1} \right] \]

(16)