

# WHY THERE IS NO IMPOSSIBILITY THEOREM ON SECURE QUANTUM BIT COMMITMENT

HORACE P. YUEN

*Department of Electrical and Computer Engineering, Department of Physics and  
Astronomy, Northwestern University, Evanston, IL 60208-3118, USA  
E-mail: yuen@ece.northwestern.edu*

The impossibility proof on unconditionally secure quantum bit commitment is critically reviewed. Different ways of obtaining secure protocols are indicated.

## 1 Introduction

There is a nearly universal acceptance of the general impossibility<sup>1-4</sup> of secure quantum bit commitment (QBC), taken to be a consequence of the Einstein-Podolsky-Rosen (EPR) type entanglement cheating which rules out QBC and other quantum protocols that have been proposed for various cryptographic objectives. Since there is no characterization of all possible QBC protocols, logically there can be no general impossibility proof as maintained to this date. In this article, which is based on Ref. 5, we explain the nature of various gaps and incompleteness in the impossibility proof, in addition to this a priori logical point. They should make clear the fact that there is no impossibility theorem even in the absence of a specific protocol that has been proved unconditionally secure. But we also describe an unconditionally secure protocol **QBC1** and other possible approaches for obtaining secure protocols.

## 2 The impossibility proof

The essential ideas that constitute the impossibility proof are generally agreed upon.<sup>1-4</sup> Adam and Babe have available to them two-way quantum communications that terminate in a finite number of exchanges, during which either party can perform any operation allowed by the laws of quantum physics. During these exchanges, Adam would have committed a bit with associated evidence to Babe. It is argued that, at the end of the commitment phase, there is an entangled pure state  $|\Phi_{\mathbf{b}}\rangle$ ,  $\mathbf{b} \in \{0, 1\}$ , shared between Adam who possesses state space  $\mathcal{H}^A$ , and Babe who possesses  $\mathcal{H}^B$ . For example, if Adam sends Babe one of  $M$  possible states  $\{|\phi_{\mathbf{b}i}\rangle\}$  for bit  $\mathbf{b}$  with probability  $p_{\mathbf{b}i}$ , then  $|\Phi_{\mathbf{b}}\rangle = \sum_i \sqrt{p_{\mathbf{b}i}} |e_i\rangle |\phi_{\mathbf{b}i}\rangle$  with orthonormal  $|e_i\rangle \in \mathcal{H}^A$  and given  $|\phi_{\mathbf{b}i}\rangle \in \mathcal{H}^B$ . Adam would open by making a measurement on  $\mathcal{H}^A$ , say  $\{|e_i\rangle\}$ , communicating to Babe his result  $i_0$  and  $\mathbf{b}$ ; then Babe would verify by measuring  $|\phi_{\mathbf{b}i_0}\rangle\langle\phi_{\mathbf{b}i_0}|$  on  $\mathcal{H}^B$ , accepting as correct only the result 1.

Generally, Babe can try to identify the bit from  $\rho_{\mathbf{b}}^B$ , the marginal state of  $|\Phi_{\mathbf{b}}\rangle$  on  $\mathcal{H}^B$ , by performing an optimal quantum measurement that yields the optimal cheating probability  $\bar{P}_c^B$  for her. Adam cheats by committing  $|\Phi_0\rangle$

and making a measurement on  $\mathcal{H}^A$  to open  $i_0$  and  $\mathbf{b} = 1$ . His probability of successful cheating is computed through  $|\Phi_{\mathbf{b}}\rangle$ , his particular measurement, and Babe's verifying measurement; the one optimized over all of his possible actions will be denoted  $\bar{P}_c^A$ . For a fixed measurement basis, Adam's cheating can be described by a unitary operator  $U^A$  on  $\mathcal{H}^A$ . When  $\rho_0^B = \rho_1^B$ , i.e.,  $\bar{P}_c^B = 1/2$ ,  $U^A$  is obtained via the Schmidt decomposition of  $|\Phi_{\mathbf{b}}\rangle$ . For unconditional, rather than perfect, security, one demands that both cheating probabilities  $\bar{P}_c^B - 1/2$  and  $\bar{P}_c^A$  can be made arbitrarily small when a security parameter  $n$  is increased.<sup>2</sup> Thus, *unconditional security* is quantitatively expressed as

$$(US) \quad \lim_n \bar{P}_c^B = \frac{1}{2}, \quad \lim_n \bar{P}_c^A = 0. \quad (1)$$

This condition (1) says that, for any  $\epsilon > 0$ , there exists an  $n_0$  such that for all  $n > n_0$ ,  $\bar{P}_c^B - 1/2 \leq \epsilon$  and  $\bar{P}_c^A \leq \epsilon$ , to which we refer as  $\epsilon$ -*concealing* and  $\epsilon$ -*binding*, respectively. These cheating probabilities are to be computed purely on the basis of physical laws, and thus would survive any change in technology, including any increase in computational power. One can write down explicitly  $\bar{P}_c^B = \frac{1}{4}(2 + \|\rho_0^B - \rho_1^B\|_1)$ . The corresponding  $\bar{P}_c^A$  satisfies:<sup>5,6</sup>

$$4(1 - \bar{P}_c^B)^2 \leq \bar{P}_c^A \leq 2\sqrt{\bar{P}_c^B(1 - \bar{P}_c^B)}. \quad (2)$$

The lower bound in (2) yields the impossibility proof<sup>1,7</sup>

$$(IP) \quad \lim_n \bar{P}_c^B = \frac{1}{2} \Rightarrow \lim_n \bar{P}_c^A = 1. \quad (3)$$

When random numbers known only to one party are used in the commitment, they are to be replaced by corresponding entanglement purification. For a random  $k$ , it is argued from the doctrine of the "Church of the Larger Hilbert Space"<sup>4</sup> that it is to be replaced by the purification  $|\Psi\rangle$  in  $\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}$ ,

$$|\Psi\rangle = \sum_k \sqrt{\lambda_k} |\psi_k\rangle |f_k\rangle, \quad (4)$$

where the  $|f_k\rangle$ 's are complete orthonormal in  $\mathcal{H}^{B_2}$  kept by Babe while  $\mathcal{H}^{B_1}$  would be sent to Adam. Similar purification is to be used for performing any operation during commitment that might otherwise require an actual measurement. As a consequence, it is claimed that a shared state  $|\Phi_{\mathbf{b}}\rangle$  at the end of commitment is known to both parties.

It appears that there is considerable incompleteness in the impossibility proof. For example, one may observe that the cheating probability  $\bar{P}_c^A$  depends on Babe's verifying measurement. For an arbitrary protocol, the impossibility proof formulation does not, and in fact cannot, specify what the possible verifying measurements could be. There is *no* proof given that there cannot be more than one verifying measurement for which different cheating transformations are needed. When such a situation occurs, Adam may not know which one to use for a successful cheating. Even though this gap can

be closed, in a proof that is not totally obvious, it is indicative of the incompleteness of the impossibility proof. The following situations show that the impossibility proof formulation is actually widely incomplete. A protocol may involve cheating detection during commitment with corresponding possibility of aborting the protocol, a situation different from cheat-sensitive protocols.<sup>8</sup> It has to be decided what would happen when cheating is detected, say in a game-theoretic formulation. It makes no sense to keep trying until one party's cheating is not detected; some limit on the number of detected cheats must be imposed. Assuming both parties are honest not trying to cheat, which is what the impossibility proof formulation does except for Adam to form entanglement instead of sending one  $|\phi_{bi}\rangle$ , also makes no sense because there would then be no need for a protocol. (Actually, the  $|\phi_{bi}\rangle$  entanglement step is often mistakenly described as an honest one.) These possibilities have not been accounted for. In the discussions of a proper framework for QBC protocols in Ref. 5, we have codified some intuitively valid rules for protocol formation under the names *Intent Principle* and *Libertarian Principle*. In the following, we will discuss several of the many gaps in the impossibility proof.

### 3 No impossibility theorem without QBC definition

A plausible first reaction to the impossibility proof is: why are all possible QBC protocols covered by its formulation? More precisely, how may one define the necessary feature of an unconditionally secure QBC protocol that is required for any proof of a mathematical theorem that says such protocol is impossible? *No* such definition is available. The situation is similar to the lack of a definition of an “effectively computable” function in the context of the Church-Turing thesis. Nobody calls the Church-Turing thesis the Church-Turing theorem. This is because there is no mathematical definition of an effectively computable function. The logical possibility is open that someday a procedure may be found that is intuitively or even physically effective, but which can compute a nonrecursive arithmetical function.

Thus in the absence of a precise definition of a QBC protocol, one would have at best an “impossibility thesis”, not an impossibility *theorem*. (This view was emphasized to the author by Masanao Ozawa.) Just as there appear to be many different forms of effective procedures, there are many different QBC protocol types<sup>5</sup> that appear not to be captured by the impossibility proof formulation. To uphold just an “impossibility thesis”, one would need to prove that unconditionally secure QBC is impossible in each of these types.

### 4 Unknown versus random parameter

The impossibility proof regards any unknown number to one party as a random variable with a known probability distribution, from which the purification (4) may be formed. However, as it is well-known in classical statistics,

not every unknown parameter is a random variable. In the present situation, there is an infinite number of open possibilities, such as the number of states and operations available, that admits no uniform probability distribution or actual entanglement for the purpose of EPR cheats. Furthermore, there is simply no ensemble here for the unknown parameter to be averaged over. In an analogous situation in the quantum information literature, this error has been recently called the “Partition Ensemble Fallacy Fallacy”.<sup>9</sup> More significantly, there is no need for Adam to know the probability  $\{\lambda_k\}$  under concealing for every  $\{\lambda_k\}$ . The proper approach is to regard the state  $|\Psi\rangle$  of (4) as an unknown “parameter” in an infinite space. The other party does not need to know it, or to know its probability distribution *even* if it has one, because of the following *Secrecy Principle* which is a corollary of the Intent Principle and Libertarian Principle.

*Secrecy Principle:* A party does not need to reveal a secret parameter chosen by her in whatever manner if it does not affect the security of the other party, who cannot reject the protocol on such a basis.

Thus, generation of the secret parameter can be automatized by one party, and it can be kept secret just as Adam can keep his bit  $b$  secret or a secret key can be kept secret in standard cryptography.

Indeed, with the use of (4) by Babe, it is not sufficient for concealing to assume that one fixed  $|\Psi\rangle$  is used by her as done in the impossibility proof. Two examples are given in Ref. 6, which show that Babe can cheat by using another  $\{\lambda_k\}$  or  $|\Psi\rangle$  than the one prescribed, and nothing in the impossibility proof formulation prevents her from doing that. If one imposes the condition that the protocol is  $\epsilon$ -concealing for every possible choice of  $|\Psi\rangle$ , then there is *no* impossibility proof until one shows that there is a cheating transformation for Adam which will work for every possible  $|\Psi\rangle$ . In the case of perfect concealing, this has been proved<sup>6</sup> for a single use of (4) by Babe. The corresponding  $\epsilon$ -concealing case is yet to be resolved. See the article by G. M. D’Ariano in this volume for a quantitative discussion.

Note that the Secrecy Principle directly contradicts the claim that a pure  $|\Phi_b\rangle$  is openly known at the end of commitment. One consequence is that because Babe does not know  $\{p_{b_i}\}$ , the usual specification of the concealing condition is a sufficient but not necessary one needed for a general impossibility proof. Furthermore, one has to show that whatever information Adam lacks on  $|\Phi_b\rangle$ , such as the  $|f_k\rangle$  of (4), is *not* needed for his cheating. Observe also that (4) is not equivalent to the mere generation of  $|\psi_k\rangle$  with probability  $\lambda_k$ , due to the presence of off-diagonal terms  $|f_k\rangle\langle f_{k'}|$ . Such purification has to be considered because of possible entanglement cheating, not because of the Church of the Larger Hilbert Space. Indeed, entanglement may help determine the bit through such terms, as the example in the next section shows. Even with the Church, the two cases are not equivalent.

## 5 Shifting of the evidence state space

Even when a pure  $|\Phi_{\mathbf{b}}\rangle$  is openly known, the impossibility proof does not cover the situations in which opening and verification are more elaborate, involving component parts of  $\mathcal{H}^A$  and  $\mathcal{H}^B$ . In particular, consider a protocol in which Babe forms (4) and sends Adam  $\mathcal{H}^{B_1}$ , with  $|\psi_k\rangle = |\psi_{k1}\rangle|\psi_{k2}\rangle$  in  $\mathcal{H}^{B_1} = \mathcal{H}^{B_{11}} \otimes \mathcal{H}^{B_{12}}$ . Adam randomly switches the state in  $\mathcal{H}^{B_{11}}$  to be that of  $|\psi_{k1}\rangle$  or  $|\psi_{k2}\rangle$  by the unitary permutation  $P_m$ ,  $m \in \{1, 2\}$ , modulates the resulting state in  $\mathcal{H}^{B_{11}}$  by a single  $U_{\mathbf{b}}$  for each  $\mathbf{b}$ , and sends it to Babe. He opens by revealing  $\mathbf{b}$ , his random permutation  $P_m$ , and *returning*  $\mathcal{H}^{B_{12}}$ . Babe verifies by testing the appropriate states in  $\mathcal{H}^{B_{11}}$  for checking  $\mathbf{b}$ , and  $\mathcal{H}^{B_{12}}$  for checking that there is no change. It is possible that the protocol is both concealing and binding for the following reason. For the final committed state  $|\Phi_{\mathbf{b}}\rangle$  with Adam entangling the  $P_m$  with  $|e_i\rangle \in \mathcal{H}^{A_1}$ , we have  $\mathcal{H}^A = \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_{12}}$  and  $\mathcal{H}^B = \mathcal{H}^{B_{11}} \otimes \mathcal{H}^{B_2}$ . Thus,  $\rho_0^B$  can be close to  $\rho_1^B$  because  $\mathcal{H}^{B_{12}}$  is not available to Babe for her cheating. However, only  $\mathcal{H}^{A_1}$ , and not  $\mathcal{H}^A$ , is available to Adam's cheating, so he cannot apply the required cheating  $U^A$  without being found cheating with a nonvanishing probability. Using the upper bound in (2) the security condition can be expressed as  $\rho_0^B(\mathcal{H}^{B_{12}} \otimes \mathcal{H}^{B_2}) \sim \rho_1^B(\mathcal{H}^{B_{12}} \otimes \mathcal{H}^{B_2})$  and  $\rho_0^B(\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}) \not\sim \rho_1^B(\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2})$ . To preserve the impossibility proof one would need to show that, in addition to (3),  $\lim_n \bar{P}_c^B(\mathcal{H}^{B_{12}} \otimes \mathcal{H}^{B_2}) = \frac{1}{2} \Rightarrow \lim_n \bar{P}_c^B(\mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2}) = \frac{1}{2}$ . Clearly, this has *not* been proved.

As an example, consider the case  $\mathcal{H}^{B_1} = \mathcal{H}^{B_{11}} \otimes \mathcal{H}^{B_{12}} \otimes \mathcal{H}^{B_{13}} \otimes \mathcal{H}^{B_{14}}$  of four qubits, with  $\{|\psi_k\rangle\} = \{|1\rangle|2\rangle|3\rangle|4\rangle, |4\rangle|1\rangle|2\rangle|3\rangle, |3\rangle|4\rangle|1\rangle|2\rangle, |2\rangle|3\rangle|4\rangle|1\rangle\}$ , where  $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$  are, e.g., a fixed set  $S_0$  of four possible BB84 states on a given great circle of a qubit. Adam permutes each  $|\psi_k\rangle$  by one of four possible  $P_m$ , and returns the first qubit to Babe unchanged for  $\mathbf{b} = 0$ , while shifted by  $\pi$  in the great circle for  $\mathbf{b} = 1$ . Assume first that Babe either did not entangle, or cannot use her entanglement in  $\mathcal{H}^{B_2}$ . Then  $\rho_0^{B_{11}}(\psi_k) = \rho_1^{B_{11}}(\psi_k)$  for all  $k$ , and no entanglement of permutations would produce a rotation on the first qubit while not disturbing the others. Thus, Adam cannot cheat perfectly and has a fixed  $\bar{P}_c^A$  for this protocol which is not arbitrarily close to one, even though it is perfectly concealing. If one can find a case in which the protocol remains perfectly concealing with entanglement by Babe, which is not the case in this example, (IP) of (3) would be contradicted, and the case can be extended to become an unconditionally secure protocol by repeating it in a sequence. Such a case can indeed be found in this class of protocols which we call Type 2.

## 6 Protocol QBC1

If carried out honestly, this protocol is conceptually simple and works as follows.<sup>5</sup> Adam sends Babe  $n$  qubits with states selected randomly and inde-

pendently from  $S_0$ . Babe then picks randomly one of these qubits and sends it back to Adam, who would leave it unchanged or shift it by  $\pi$ , depending on whether  $\mathbf{b} = 0$  or  $1$ , and commit it as evidence. He opens by revealing  $\mathbf{b}$  and all the qubit states, and Babe verifies by corresponding measurements.

We assume that no cheating by either party, other than entanglement, occurs during commitment as in the impossibility proof formulation, say, under heavy penalty in a game-theoretic formulation in which state checking is done by both parties. Thus the protocol is perfectly concealing. There are many ways for Babe to randomly pick one of the  $n$  qubits, say by permutation into a fixed qubit among the  $n$  ones, or into a separate fixed qubit, each with its own purification. If Adam knows which particular way Babe chooses, it can be shown that he can cheat successfully. However, his success depends crucially on this knowledge, and no further entanglement purification by Babe is possible over these different ways that would allow her to send back a single qubit to Adam for bit modulation. While the situation here has some similarity to our Type 3 protocols,<sup>5</sup> it is one that cannot be completely purified even with a known probability distribution, and the impossibility proof does not apply. Thus, the protocol becomes  $\epsilon$ -binding for large  $n$ . A full security proof of this protocol and detailed treatment of Type 2 protocols will be presented elsewhere.

### Acknowledgements

I would like to thank G. M. D'Ariano, M. Ozawa, and I. L. Chuang for useful discussions. This work was supported by the Defense Advanced Research Projects Agency and the Army Research Office.

### References

1. D. Mayers, quant-ph/9603015.
2. D. Mayers, Phys. Rev. Lett. **78**, 3414 (1997).
3. H. K. Lo and H. F. Chau, Phys. Rev. Lett. **78**, 3410 (1997); Fortschr. Phys. **46**, 907 (1998); Physica D **120**, 177 (1998); H. K. Lo, Phys. Rev. A **56**, 1154 (1997); G. Brassard, C. Crépeau, D. Mayers, and L. Salvail, quant-ph/9712023; quant-ph/9806031; J. Mueller-Quade and H. Imai, quant-ph/0010112.
4. D. Gottesman and H. K. Lo, Physics Today, Nov. 2000, p. 22.
5. H. P. Yuen, quant-ph/0207089.
6. H. P. Yuen, quant-ph/0109055.
7. H. P. Yuen, quant-ph/0006109.
8. L. Hardy and A. Kent, quant-ph/9911043.
9. K. Nemoto and S. L. Braunstein, quant-ph/0207135.