

# ENTROPY-ENERGY BALANCE IN NOISY QUANTUM COMPUTERS

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We use entropy-energy arguments to assess the limitations on the running time and on the system size, as measured in qubits, of noisy macroscopic circuit-based quantum computers.

Quantum information is essentially a nonequilibrium theory in the sense that its basic primitives are decomposed into temporal sequences of local operations that are performed on physical systems driven far from equilibrium. The requirement of locality becomes all the more important when we consider quantum information processing in macroscopic systems (multi-qubit quantum computers), because local operations will be used to model both the computation proper and the errors affecting it.

An important issue to address is the stability of a macroscopic quantum computer (MQC) in the presence of noise. There are two notions of stability for quantum-mechanical states,<sup>1</sup> namely *global* and *local thermodynamic stability*. For a microscopic system (i.e., one with finitely many degrees of freedom) these are equivalent and amount to the following. Let  $H$  be the system Hamiltonian,  $\beta$  the inverse temperature. Then a state (density operator)  $\rho$  is thermodynamically stable (both globally and locally) if it minimizes the free energy functional  $F_\beta(\rho) = \text{tr}(\rho H) - (1/\beta)S(\rho)$ , where  $S(\rho) = -\text{tr}(\rho \ln \rho)$  is the von Neumann entropy. However, the two stability notions are rather different for systems with infinitely many degrees of freedom. A state  $\rho$  is globally thermodynamically stable (GTS) if it minimizes the specific free energy (i.e., free energy “per particle”), and locally thermodynamically stable (LTS) if no *local* modification of it yields a state with lower specific free energy. It can be shown that any GTS state is also LTS, but the converse is generally false.

States that are LTS but not GTS are referred to as *metastable states*.<sup>1</sup> An apt example comes from laser physics. Imagine an active medium consisting of a large number of three-level atoms with the states  $|0\rangle, |1\rangle, |2\rangle$ , each atom initially in the ground state  $|0\rangle$ . The atoms are pumped to the level  $|2\rangle$ , and then decay nonradiatively to  $|1\rangle$ . This results in a relatively long-lived metastable state of population inversion. As far as MQC’s are concerned, we are interested mainly in their metastable states, and the goal of error correction (and of judicious MQC design in general) is to preserve metastability for the duration of the computation. Specifically, we can isolate two separate aspects of (meta)stability for MQC’s — temporal and spatial. The former

refers to the maximum number of operations that can be carried out on a noisy MQC using given energy resources, whereas the latter is related to the maximum number of qubits that can be processed reliably on a noisy MQC. We will address these two issues using entropy-energy arguments,<sup>2</sup> a standard technique in statistical physics. Due to lack of space, our presentation will be rather sketchy; we will supply the details in a separate paper.<sup>3</sup> We note that even though we consider *macroscopic systems*, there is no need to pass to the thermodynamic limit because we are concerned with *local* thermodynamic stability.

First of all, let us fix a model of an  $N$ -qubit quantum computer. We may imagine its operation as a sequence of alternating computation and noise steps. For a circuit-based QC, a computation step is an application of a tensor product of one- and two-qubit universal quantum gates, and a noise step is described by a general trace-preserving completely-positive linear map  $T$  on density matrices. We will henceforth refer to such maps as channels (note that any unitary transformation is an instance of a channel). Specifically, if  $\mathcal{H} \simeq \mathbb{C}^2$  is a single-qubit Hilbert space, then the state of the  $N$ -qubit computer is a density operator  $\rho$  on  $\mathcal{H}^{\otimes N}$ . The composition of a computation step and a noise step is a mapping  $\rho \mapsto T(U\rho U^*)$ , where  $U$  is unitary. We adopt the model of local stochastic noise,<sup>4</sup> so that  $T$  can be written in the form  $(1 - \epsilon)\text{id} + \epsilon R^{\otimes N}$ , where  $\text{id}$  is the identity channel,  $R$  is a channel on  $2 \times 2$  density matrices, and  $\epsilon$  is a small positive number that quantifies the noise strength.

We make two simplifying assumptions. The first one is needed for the analysis of the temporal stability and concerns the noisy channel  $R$ . Namely, we will take  $R$  to be bistochastic, i.e.,  $R(\mathbb{1}) = \mathbb{1}$ , and strictly contractive,<sup>5</sup> which means that there exists a constant  $C \in [0, 1)$  (called the *contraction rate*) such that, for any two density operators  $\rho, \sigma$ ,  $\|R(\rho) - R(\sigma)\|_1 \leq C \|\rho - \sigma\|_1$ , where  $\|X\|_1 = \text{tr}|X|$  is the trace norm. For instance, the much studied depolarizing channel,  $R(\rho) = (1 - \lambda)\rho + \lambda\mathbb{1}/2$  for some  $\lambda \in [0, 1)$ , fits this description. Our second assumption, necessary only for the analysis of the spatial stability, states that the number of gates applied during any computation step is bounded by a positive number  $K$  that depends only on the particular algorithm, but not on the number of qubits  $n$ . This means that our analysis will be inapplicable to highly parallelized computation.<sup>6</sup>

We will rely upon the following two theorems, the proofs of which are given elsewhere.<sup>3,7</sup>

**Theorem 1.** Consider the channel  $R_N = R^{\otimes N}$ , where  $R$  is a bistochastic strictly contractive channel on  $2 \times 2$  density matrices with contraction rate  $C$ . Then, for any  $2^N \times 2^N$  density matrix  $\rho$  and any positive integer  $m$ , we have  $S[R_N^m(\rho)] - S(\rho) \geq \frac{1-C^{2m}}{2} \|\rho - 2^{-N}\mathbb{1}\|_2^2$ , where  $S(\rho) = -\text{tr}(\rho \ln \rho)$  is the von Neumann entropy and  $\|X\|_2 = \sqrt{\text{tr}(X^*X)}$  is the Hilbert-Schmidt

norm of the operator  $X$ .

**Theorem 2.** Let  $\rho_i$ ,  $i = 1, \dots, n$ , be  $n$  mutually commuting density operators. Suppose that there exists a constant  $\kappa \geq 0$  such that, for any  $i$ ,  $\sum_{j \neq i} \text{tr}(\rho_j \rho_i) \leq \kappa$ . Let  $\rho = n^{-1} \sum_{i=1}^n \rho_i$ . Then  $S(\rho) \geq n^{-1} \sum_{i=1}^n S(\rho_i) + \ln(n) - 2\sqrt{\kappa}$ .

Let us first consider temporal stability. Suppose that our computer operates on  $N$  qubits, and that we are given “energy resources”  $E$  (this parameter could be determined, e.g., from the so-called “control Hamiltonian” representation of quantum computation<sup>8</sup>). We are interested in the entropy increase after  $m$  steps of noisy computation. Since entropy can only be produced by the channel  $T$  and not by the unitarily implemented gates, the entropy gain is given by  $\Delta S(\rho, m) \equiv S[T^m(\rho)] - S(\rho)$ , where  $\rho$  is the initial state of the computer. Now  $T^m = \sum_{j=0}^m \binom{m}{j} (1 - \epsilon)^{m-j} \epsilon^j R_N^j$ , where  $R_N = R^{\otimes N}$ , so that by concavity of the von Neumann entropy we have  $\Delta S(\rho, m) \geq \sum_{j=0}^m \binom{m}{j} (1 - \epsilon)^{m-j} \epsilon^j \{S[R_N^j(\rho)] - S(\rho)\}$ . Using Theorem 1 we can write

$$\Delta S(\rho, m) \geq \zeta_{\rho, N} \sum_{j=0}^m \binom{m}{j} (1 - \epsilon)^{m-j} \epsilon^j (1 - C^{2j}), \quad (1)$$

where  $\zeta_{\rho, N} \in [0, (2^N - 1)/2^{N+1}]$  depends only on  $\rho$  and  $N$ . Carrying out the summation in (1) we obtain

$$\Delta S(\rho, m) \geq \zeta_{\rho, N} \{1 - [1 - \epsilon(1 - C^2)]^m\} = \zeta_{\rho, N} m \epsilon (1 - C^2) + o(\epsilon), \quad (2)$$

where  $o(\epsilon)$  stands for terms that go to zero faster than  $\epsilon$  as  $\epsilon \rightarrow 0$  and therefore can be neglected when  $\epsilon$  is sufficiently small. The corresponding free-energy shift is  $\Delta F_\beta(\rho, m) \equiv E - (1/\beta)\Delta S(\rho, m)$ , where  $\beta$  is the inverse temperature. It is now easy to see from the estimate (2) that  $\Delta F_\beta(\rho, m)$  will be negative for all  $m \geq \bar{\zeta}_{\rho, N, T} \beta E$ , where  $\bar{\zeta}_{\rho, N, T}$  is a constant that depends on  $\rho$ ,  $N$ , and  $T$ . In other words, the state of the MQC will not be metastable for large enough  $m$ , or, to put it bluntly, the MQC will “crash” after  $O(\beta E)$  operations. However, because the produced entropy may be at least partially drained off by error correction, this estimate merely tells us how often it should be carried out.

We pass now to spatial stability. We note that in a circuit-based computer each computational step is an application of a bounded number of universal quantum gates, and the initial state of the computer may be taken separable without loss of generality. Therefore at every instant of the computation the set  $\{1, \dots, N\}$  can be partitioned into  $L$  disjoint clusters  $C_1, \dots, C_L$ , so that the state of the computer has the form  $\rho = \bigotimes_{l=1}^L \rho^{(l)}$ , where  $\rho^{(l)}$  is the state of the qubits in the  $l^{\text{th}}$  cluster.<sup>4</sup> Let us suppose for simplicity that the clusters all have the same size  $d$ , so that the computer operates on  $N = Ld$  qubits. Let us partition the set  $\{C_1, \dots, C_L\}$  into  $n$  disjoint blocks  $S_1, \dots, S_n$ , each of which

contains  $k$  clusters; thus  $N = nkd$ . Now, for each  $i = 1, \dots, n$ , define  $T_i$  to be the channel that acts as a depolarizing channel  $\sigma \mapsto (1 - \lambda)\sigma + \lambda\mathbb{1}/2^d$  on the state of each of the clusters in  $S_i$  and leaves the remaining clusters intact. Then  $\rho_i \equiv T_i(\rho)$  are  $n$  mutually commuting density operators for which we have the elementary estimate

$$\mathrm{tr} \left( \sum_{j \neq i} \rho_j \rho_i \right) \leq (n-1) \left( 1 - \lambda + \frac{\lambda}{2^d} \right)^{2k}. \quad (3)$$

Note that for fixed  $n$ ,  $\lambda$ , and  $d$  we can always choose  $k$  so large that the right-hand side of (3) is bounded above by unity. Thus the conditions of Theorem 2 are satisfied with  $\kappa = 1$ , and the entropy gain due to the channel  $T \equiv n^{-1} \sum_{i=1}^n T_i$  is  $O(\ln n)$ . Because the energy shift due to  $T$  depends only on the number  $K$  of gates applied during each computational step, the free-energy shift will be negative for large enough  $n$ . We may thus conclude that the number of qubits that can be processed reliably (that is, kept in a metastable state) in a MQC is bounded from above as  $O(e^{\beta E})$ , where  $\beta$  is the inverse temperature, and  $E$  depends on  $K$ .

To summarize, we have shown that there exist upper bounds on the running time and on the size (in qubits) of circuit-based noisy MQC's. However, these constraints become significant only when  $\beta E$  is small (as in, e.g., ensemble quantum computation using NMR). Furthermore, our analysis applies only to circuit-based computers without parallelization; massively parallel non-circuit models, such as the ‘‘one-way quantum computer’’ of Briegel and Raussendorf,<sup>9</sup> are likely to be intrinsically thermodynamically stable.

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