FUNDAMENTAL LIMITATION ON QUBIT OPERATIONS DUE TO THE BLOCH-SIEGERT OSCILLATION

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We show that if the Rabi frequency is comparable to the Bohr frequency, so that the rotating wave approximation is inappropriate, an extra oscillation is present with the Rabi oscillation. We discuss how the sensitivity of the degree of excitation to the phase of the field may pose severe constraints on precise rotations of quantum bits involving low-frequency transitions. We present a scheme for observing this effect in an atomic beam.

It is well known that the amplitude of an atomic state is necessarily complex valued. The electric or magnetic field generated by an oscillator, on the other hand, is real valued, composed of the sum of two complex components. In describing semiclassically the atom-field interaction involving such a field, one often side-steps this difference by making the so-called rotating wave approximation (RWA), under which only one of the two complex components is kept, and the counterrotating part is ignored. We discuss how the sensitivity of the degree of excitation to the phase of the field poses severe constraints on precise rotations of quantum bits involving low-frequency transitions. We also present a scheme for observing this effect in an atomic beam.

We consider an ideal two-level system in which a ground state $|0\rangle$ is coupled to an excited state $|1\rangle$, and $0 \leftrightarrow 1$ transitions are magnetic dipolar with transition frequency ω, and the magnetic field is of the form $B = B_0 \cos(\omega t + \phi)$. We now summarize briefly the two-level dynamics without the RWA. In the dipole approximation, the Hamiltonian can be written as:

$$\hat{H} = \epsilon(\sigma_0 - \sigma_z)/2 + g(t)\sigma_x, \tag{1}$$

where $g(t) = -g_0 \left[\exp(i\omega t + i\phi) + \text{c.c.}\right]/2$, σ_i are Pauli matrices, and $\epsilon = \omega$ corresponds to resonant excitation. The state vector is written as:

$$|\xi(t)\rangle = \begin{pmatrix} C_0(t) \\ C_1(t) \end{pmatrix}. \tag{2}$$

We perform a rotating wave transformation by operating on $|\xi(t)\rangle$ with the unitary operator $\hat{Q}$, where: $\hat{Q} = (\sigma_0 + \sigma_z)/2 + \exp(i\omega t + i\phi)(\sigma_0 - \sigma_z)/2$. The Schrödinger equation then takes the form (setting $\hbar = 1$): $|\dot{\tilde{\xi}}\rangle = -i\tilde{H(t)}|\tilde{\xi}(t)\rangle$, where the effective Hamiltonian is given by

$$\tilde{H} = \alpha(t)\sigma_+ + \alpha^*(t)\sigma_-, \tag{3}$$

with $\alpha(t) = -(g_0/2)\,[\exp(-i2\omega t - i2\phi) + 1]$, and in the rotating frame the state vector is

$$|\tilde{\xi}(t)\rangle \equiv \hat{Q}|\xi(t)\rangle = \begin{pmatrix} \tilde{C}_0(t) \\ \tilde{C}_1(t) \end{pmatrix}. \tag{4}$$

Now, by making the rotating wave approximation (RWA), corresponding to dropping the fast oscillating term in $\alpha(t)$, one ignores effects (such as the Bloch-Siegert shift) of the order of (g_0/ω), which can easily be observed in experiment if g_0 is large.[1,2] Otherwise, by choosing g_0 to be small enough, one can make the RWA for any value of ω. We explore here both regimes, and we find the general results without the RWA.

From Eqs. (3) and (4), we get two coupled differential equations:

$$\dot{\tilde{C}}_0(t) = i(g_0/2)\,[1 + \exp(-i2\omega t - i2\phi)]\,\tilde{C}_1(t) \tag{5}$$

$$\dot{\tilde{C}}_1(t) = i(g_0/2)\,[1 + \exp(+i2\omega t + i2\phi)]\,\tilde{C}_0(t). \tag{6}$$

We choose $|C_0(0)|^2 = 1$ for the initial condition, and proceed to find approximate analytical solutions of Eqs. (5) and (6). Due to the periodic nature of the effective Hamiltonian, the general solution to Eqs. (5) and (6) can be written in the form:

$$|\tilde{\xi}(t)\rangle = \sum_{n=-\infty}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \exp[n(-i2\omega t - i2\phi)]. \tag{7}$$

Inserting Eq. (7) in Eqs. (5) and (6) we get for all n :

$$\dot{a}_n = i2n\omega a_n + ig_0(b_n + b_{n-1})/2, \tag{8}$$

$$\dot{b}_n = i2n\omega b_n + ig_0(a_n + a_{n+1})/2. \tag{9}$$

Here, the coupling between a_0 and b_0 is the conventional one when the RWA is made. The couplings to the nearest neighbors, $a_{\pm 1}$ and $b_{\pm 1}$, are detuned by an amount 2ω, and so on. To lowest order in (g_0/ω), we can ignore terms with $|n| > 1$, thus yielding a truncated set of six equations for $\dot{a}_0, \dot{b}_0, \dot{a}_{\pm 1}, \dot{b}_{\pm 1}$. Now consider g_0 to have a time dependence of the form $g_0(t) = g_{0M}\,[1 - \exp(-t/\tau_{sw})]$, where the switching time constant τ_{sw} is large compared to other characteristic time scales such as $1/\omega$ and $1/g_{0M}$. Then, these equations can be solved by the method of adiabatic elimination, which is valid to first order in $\eta \equiv (g_0/4\omega)$. [Note $\eta(t) = \eta_0\,[1 - \exp(-t/\tau_{sw})]$, where $\eta_0 \equiv (g_0M/4\omega)$]. To solve the set of equations above, we consider first a_{-1} and b_{-1}. Define $\mu_\pm \equiv (a_{-1} \pm b_{-1})$, then one can write $\dot{\mu}_\pm = -i(2\omega + g_0/2)\mu_\pm \pm ig_0 a_0/2$. Adiabatic following then yields (again, to lowest order in η) $a_{-1} \approx 0$ and $b_{-1} \approx \eta a_0$. Likewise, we can show that $a_1 \approx -\eta b_0$ and $b_1 \approx 0$. The only nonvanishing (to lowest order in η with $|C_0(t = 0)|^2 = 1$) terms in the solution of Eqs. (5) and (6) are:

$$C_0(t) = \cos(g_0'(t)t/2) - i\eta e^{-i(2\omega t + 2\phi)} \sin(g_0'(t)t/2), \tag{10}$$

$$C_1(t) = ie^{-i(\omega t + \phi)}[\sin(g_0'(t)t/2) - i\eta e^{+i(2\omega t + 2\phi)} \cos(g_0'(t)t/2)], \tag{11}$$

where $g_0'(t) = 1/t \int_0^t g_0(t)dt = g_0 \left[1 - (t/\tau_{sw})^{-1} \exp(-t/\tau_{sw})\right]$.

To lowest order in η, this solution is normalized at all times. Note that if one applies this excitation to an ensemble of atoms using a $\pi/2$-pulse and then measures the population of state $|1\rangle$ after the excitation terminates (at $t = \tau$, when $g'(\tau)\tau/2 = \pi/2$), the output signal will be

$$|C_1(g_0'(\tau), \phi)|^2 = \frac{1}{2} \left[1 + 2\eta \sin(2\omega\tau + 2\phi)\right], \tag{12}$$

which contains information about both the amplitude and the phase of the driving field B. This clearly indicates that the Rabi transition probability depends on the total phase $\phi_\tau = \omega\tau + \phi$ of the driving field.

A physical realization of this result can be best appreciated by consider-

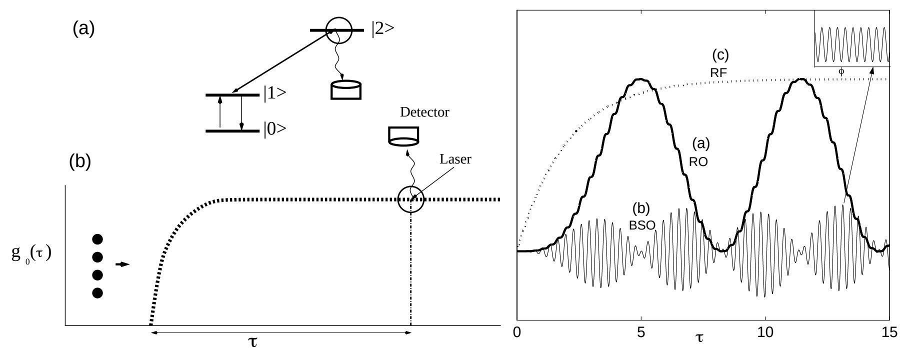

Figure 1. Left: Schematic illustration of an experimental arrangement for measuring the phase dependence of the population of the excited state $|1\rangle$: (a) The microwave field couples the ground state ($|0\rangle$) to the excited state ($|1\rangle$). A third level, $|2\rangle$, which can be coupled to $|1\rangle$ optically, is used to measure the population of $|1\rangle$ via fluorescence detection. (b) The microwave field is turned on adiabatically with a switching time-constant τ_{sw}, and the fluorescence is monitored after a total interaction time of τ. Right: Illustration of the Bloch-Siegert Oscillation (BSO): (a) The population of state $|1\rangle$, as a function of the interaction time τ , showing the BSO superimposed on the conventional Rabi oscillation. (b) The BSO oscillation (amplified scale) by itself, produced by subtracting the Rabi oscillation from the plot in (a). (c) The time-dependence of the Rabi frequency. Inset: BSO as a function of the absolute phase of the field with fixed τ.

ing an experimental arrangement of the type illustrated in the left panel of Fig. 1; a plot of the associated Rabi oscillations is shown in the right panel of Fig. 1. Under the RWA, curve (a) on the right in Fig. 1 represents the conventional Rabi oscillation. However, we notice here some additional oscillation—magnified and shown separately in curve (b) of this panel—that is produced by subtracting the conventional Rabi oscillation, $\sin^2(g(t)/2)$, from curve (a). That is, curve (b) in the right panel of Fig. 1 corresponds to what we call the Bloch-Siegert Oscillation (BSO), given by $\eta \sin(g_0'(\tau)\tau) \sin(2\phi_\tau)$. These analytical results agree very closely with the results obtained via direct numerical integration of Eqs. (5) and (6). Note that the BSO is at twice the frequency of the driving field, and its amplitude vanishes when all the atoms are in a single state. This oscillation will be stronger when the ratio of the resonance frequency to the Rabi frequency is large.[3,4,5] One should keep track of the phase of the excitation field at the location of the qubit[6,7] for low-energy qubit systems[8,9,10] with fast driving fields.

In conclusion, we have shown that when a two-level atomic system is driven by a strong periodic field, the Rabi oscillation is accompanied by another oscillation at twice the transition frequency. This extra oscillation can limit the qubit operations in Rabi flopping without a proper matching of the parameters. However, it has been shown that this phase has potential application in teleportation. By making use of distant entanglement, this mechanism may enable teleportation of the phase of a field that is encoded in the atomic state amplitude, and has potential applications to remote frequency-locking.[11,12,13,14]

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