Characterizing the entanglement of bipartite quantum systems

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We derive a separability criterion for bipartite quantum systems which generalizes the already known criteria. It is based on observables having generic commutation relations. We then discuss in detail the relation among these criteria.

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I. INTRODUCTION

Entangled states have been known almost from the very beginning of quantum mechanics [1,2], and their somewhat unusual features have been investigated for many years. However, recent developments in the theory of quantum information [3] have required a deeper knowledge of their properties.

The simplest system where one can study entanglement is represented by a bipartite system. In such a system, either with discrete or continuous variables, the inseparability of pure states is now well understood and the von Neumann entropy of either subsystem quantifies the amount of entanglement [4]. Instead, the question of inseparability of mixed states is much more complicated and involves subtle effects. For discrete variable systems the Peres-Horodecki theorem [5] constitutes a theoretical tool to investigate the separability. Recently, different criteria have been also proposed for continuous variable systems [6–11]. Nevertheless, a unifying criterion, of practical use, does not exist yet. Needless to say that also the entanglement quantification for mixed states is not well assessed [12]. On the other hand, the lack of knowledge for bipartite entanglement is not only a serious drawback in the study of mixed-state entanglement, but also a limitation for understanding multipartite entanglement.

The aim of this paper is to throw some light on the plethora of entanglement criteria for bipartite systems. In particular, we shall derive a general separability criterion valid for any state of any bipartite system. We shall then discuss its relation with the already known criteria.

II. A GENERAL SEPARABILITY CRITERION

Let us consider a bipartite system whose subsystems, not necessarily identical, are labeled 1 and 2, and a separable state \( \hat{\rho}_{\text{sep}} \) on the Hilbert space \( \mathcal{H}_{\text{tot}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). Such a state can be written as

\[
\hat{\rho}_{\text{sep}} = \sum_k w_k \hat{\rho}_{k1} \otimes \hat{\rho}_{k2},
\]

where \( \hat{\rho}_{kj} \) \((j = 1, 2)\) are normalized density matrices on \( \mathcal{H}_j \) while \( w_k \geq 0 \) with \( \sum_k w_k = 1 \).

Let us now choose a generic couple of observables for each subsystem, say, \( \hat{r}_j, \hat{s}_j \) on \( \mathcal{H}_j \) \((j = 1, 2)\), and define the operators

\[
\hat{c}_j = i[\hat{r}_j, \hat{s}_j], \quad j = 1, 2.
\]

Notice that the two couples \( \hat{r}_1, \hat{s}_1 \) may represent completely different observables, e.g. one couple may refer to a continuous variable subsystem while the other may refer to a discrete variable subsystem. Furthermore, \( \hat{c}_j \) is typically a non-trivial Hermitian operator on the Hilbert subspaces.

We now introduce the following observables on \( \mathcal{H}_{\text{tot}} \):

\[
\hat{u} = a_1 \hat{r}_1 + a_2 \hat{r}_2, \quad \hat{v} = b_1 \hat{s}_1 + b_2 \hat{s}_2,
\]

where \( a_j, b_j \) are real parameters. From the standard form of the uncertainty principle [13], it follows that every state \( \hat{\rho} \) on \( \mathcal{H}_{\text{tot}} \) must satisfy

\[
|\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle| \geq \frac{|a_1 b_1 \langle \hat{c}_1 \rangle + a_2 b_2 \langle \hat{c}_2 \rangle|^2}{4},
\]

where \( \langle \hat{\Theta} \rangle = \text{Tr}(\hat{\Theta} \hat{\rho}) \) is the expectation value over \( \hat{\rho} \) of the operator \( \hat{\Theta} \), and \( \Delta \hat{\Theta} = \hat{\Theta} - \langle \hat{\Theta} \rangle \). However, a stronger bound exists for separable states. As a matter of fact, the following theorem holds:

Theorem.

\[
\hat{\rho}_{\text{sep}} \Rightarrow |\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle| \geq \tilde{\Theta}^2,
\]

with

\[
\tilde{\Theta} = \frac{1}{2} (|a_1 b_1| \tilde{\Theta}_1 + |a_2 b_2| \tilde{\Theta}_2),
\]

where

\[
\tilde{\Theta}_j = \sum_k w_k \langle |\hat{c}_j\rangle_k, \quad j = 1, 2,
\]

whose \( \langle \hat{\Theta}_j \rangle_k = \text{Tr}(\hat{\Theta}_j \hat{\rho}_{kj}) \) is the expectation value of the operator \( \hat{\Theta}_j \) onto \( \hat{\rho}_{kj} \).
Proof. From the definitions of \( \langle (\Delta \hat{u})^2 \rangle \) and \( \rho_{separable} \) it is easy to see that

\[
\langle (\Delta \hat{u})^2 \rangle = \sum_k w_k [a_j^2 \langle (\Delta \hat{r}_j)^2 \rangle_k + a_k^2 \langle (\Delta \hat{r}_k)^2 \rangle_k]
\]

\[
+ \sum_k w_k \langle \hat{u}_k \rangle_k^2 \left( \sum_k w_k \langle \hat{u}_k \rangle_k^2 \right)^2,
\]

where the quantity \( \Delta \hat{r}_j \equiv \hat{r}_j - \langle \hat{r}_j \rangle_k \) gives the variance of the operator \( \hat{r}_j \) on the state \( \rho_{kj} \). An analoguous expression holds for \( \langle (\Delta \hat{v})^2 \rangle \). By applying the Cauchy-Schwartz inequality on the last two terms of the right-hand side of Eq. (8) we obtain

\[
\langle (\Delta \hat{u})^2 \rangle \geq \sum_k w_k [a_j^2 \langle (\Delta \hat{r}_j)^2 \rangle_k + a_k^2 \langle (\Delta \hat{r}_k)^2 \rangle_k].
\] (9)

and analogously

\[
\langle (\Delta \hat{v})^2 \rangle \geq \sum_k w_k [b_j^2 \langle (\Delta \hat{s}_j)^2 \rangle_k + b_k^2 \langle (\Delta \hat{s}_k)^2 \rangle_k].
\] (10)

Now, taking any two real non-negative numbers \( \alpha \) and \( \beta \), and using relations (9) and (10), we get the following inequality:

\[
\alpha \langle (\Delta \hat{u})^2 \rangle + \beta \langle (\Delta \hat{v})^2 \rangle
\]

\[
\geq \sum_k w_k [\alpha a_j^2 \langle (\Delta \hat{r}_j)^2 \rangle_k + \beta b_j^2 \langle (\Delta \hat{s}_j)^2 \rangle_k]
\]

\[
+ \sum_k w_k [\alpha a_k^2 \langle (\Delta \hat{r}_k)^2 \rangle_k + \beta b_k^2 \langle (\Delta \hat{s}_k)^2 \rangle_k].
\] (11)

Furthermore, by applying the uncertainty principle to the operators \( \hat{r}_j \) and \( \hat{s}_j \) on the state \( \rho_{kj} \), it follows

\[
\alpha a_j^2 \langle (\Delta \hat{r}_j)^2 \rangle_k + \beta b_j^2 \langle (\Delta \hat{s}_j)^2 \rangle_k
\]

\[
\geq \alpha a_j^2 \langle (\Delta \hat{r}_j)^2 \rangle_k + \beta b_j^2 \frac{|\langle \hat{C}_j \rangle|}{4 \langle (\Delta \hat{r}_j)^2 \rangle_k}
\]

\[
\geq \sqrt{\alpha \beta} |a_j b_j| |\langle \hat{C}_j \rangle|.
\] (12)

The last inequality of Eq. (12) comes from the behavior, for \( x \geq 0 \), of the function

\[
f(x) = \gamma_1 x + \gamma_2 lnx
\] (13)

with \( \gamma_1, \gamma_2 \geq 0 \). Such a function takes the minimum value \( f_{\text{min}} = 2 \sqrt{\gamma_1 \gamma_2} \). Then, inserting Eq. (12), into (11) we obtain

\[
\alpha \langle (\Delta \hat{u})^2 \rangle + \beta \langle (\Delta \hat{v})^2 \rangle \geq 2 \sqrt{\alpha \beta} \mathcal{O},
\] (14)

where \( \mathcal{O} \) has been defined in Eq. (6).

Notice that, for a given system state, the quantities \( \langle (\Delta \hat{u})^2 \rangle \), \( \langle (\Delta \hat{v})^2 \rangle \), and \( \mathcal{O} \) are fixed, and the inequality (14) must be satisfied for every positive value of \( \alpha \) and \( \beta \). Thus, we can write

\[
\langle (\Delta \hat{u})^2 \rangle \geq \max_{\alpha > 0; \beta > 0} \left\{ 2 \sqrt{\frac{\beta}{\alpha}} \mathcal{O} - \frac{\beta}{\alpha} \langle (\Delta \hat{v})^2 \rangle \right\} = \frac{\mathcal{O}^2}{\langle (\Delta \hat{v})^2 \rangle},
\] (15)

where the equality has been obtained by maximizing the function \( g(x) = 2x \mathcal{O} - x^2 \langle (\Delta \hat{v})^2 \rangle \) over \( x \geq 0 \). This concludes the proof of Eq. (5).

Practically, proving the theorem, we have created a family of linear inequalities (14), which must be always satisfied by separable states. The “convolution” of such relations gives the condition (5), representable by a region in the \( \langle (\Delta \hat{u})^2 \rangle, \langle (\Delta \hat{v})^2 \rangle \) plane delimited by an hyperbola (see Fig. 1). Notice also that, since \( \mathcal{O} = \Sigma_k w_k |\langle \hat{C}_j \rangle_k| = |\Sigma_k w_k \langle \hat{C}_j \rangle_k| = |\langle \hat{C}_j \rangle| \), the following inequalities hold:

\[
\mathcal{O} \geq \frac{1}{2} |a_j b_j| |\langle \hat{C}_j \rangle_1| + |a_k b_k| |\langle \hat{C}_j \rangle_2|
\] (16)

\[
\Rightarrow \frac{1}{2} |a_j b_j| |\langle \hat{C}_j \rangle_1| + |a_k b_k| |\langle \hat{C}_j \rangle_2|.
\] (17)

In particular, Eq. (17) tells us that the bound (5) for separable states is much stronger than Eq. (4) for generic states. Moreover, Eq. (16) gives us a simple separability criterion. In fact, while \( \mathcal{O} \) is not easy to evaluate directly, as it depends on the type of convex decomposition (1) one is considering, the right hand side of Eq. (16) is easily measurable, as it depends on the expectation value of the observables \( \hat{C}_j \). In this sense we can claim that Eq. (5) is a necessary criterion for separability, i.e.,

\[
\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle < \mathcal{O}^2 \Rightarrow \rho \text{ entangled.}
\] (18)

An important simplification applies when the observable \( \hat{C}_j \) is proportional to the identity operator (e.g., \( \hat{r}_j \) is the position and \( \hat{s}_j \) is the momentum operator of a particle), or more generally when it is positive (or negative) definite. In this
case the inequality (16) reduces to an identity and the quantity \( \mathcal{O} \) does not depend on the convex decomposition (1).

The criterion (5) can be further generalized if one adopts the strong version of the uncertainty principle [13] in deriving the inequality (12). In this situation the quantity \( \tilde{\mathcal{O}}_j \) of Eq. (6) becomes

\[
\tilde{\mathcal{O}}_j = 2 \sum_k w_k |\Delta_{\hat{r}}^{(k)}(\Delta \hat{\rho})_k|, \quad j = 1,2.,
\]

where \( \Delta_{\hat{r}}^{(k)} \) and \( \Delta \hat{\rho} \) are the same objects we have introduced in Eqs. (9) and (10). Also in this case \( \tilde{\mathcal{O}} \) depends, in general, on the convex decomposition (1) of the state \( \hat{\rho} \).

III. RELATION WITH OTHER CRITERIA

In this section we analyze the relation between criterion (5) and other necessary criteria for separability that have been proposed in the past.

First of all, it is possible to show that the “sum” criterion of Ref. [9] represents a particular case of Eq. (5). As a matter of fact the sum criterion is given by Eq. (14) with \( \alpha = \beta = 1 \),

\[
\langle (\Delta \hat{u})^2 \rangle + \langle (\Delta \hat{v})^2 \rangle \geq 2 \bar{\mathcal{O}} \geq |a_1 b_1| |\langle \hat{c}_1 \rangle| + |a_2 b_2| |\langle \hat{c}_2 \rangle|.
\]

(20)

where we have exploited Eq. (16) to get a rhs independent of the convex decomposition of \( \hat{\rho}_{sep} \). The fact that the sum criterion comes from condition (5) is a consequence of the fact that the latter has been derived by maximizing over the family of inequalities (14) [see Eq. (15) and Fig. 1]. However, a straightforward derivation is easy to obtain as well. In fact, from Eq. (5) we have

\[
\langle (\Delta \hat{u})^2 \rangle + \langle (\Delta \hat{v})^2 \rangle \geq \langle (\Delta \hat{\rho})^2 \rangle \geq 2 \bar{\mathcal{O}},
\]

(21)

where, for the second inequality, we have used the property of \( f(x) \) in Eq. (13).

Let us now compare the criterion developed in Sec. II with the “product” criterion developed in Ref. [11]. The latter, with the generic operators \( \hat{u} \) and \( \hat{v} \) of Eq. (3), can be written as

\[
\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle \leq \langle (\Delta \hat{\rho})^2 \rangle \Rightarrow \left| a_1 a_2 b_1 b_2 \right| \frac{|\langle \hat{c}_1 \hat{c}_2 \rangle|^2}{||\hat{c}_1|| ||\hat{c}_2||} \geq 2 \bar{\mathcal{O}},
\]

(22)

where

\[
||\hat{c}_j|| = \sup_{|\phi| = 1} |\langle \phi | \hat{c}_j | \psi \rangle|
\]

(23)

is the norm of the operator \( \hat{c}_j \) (\( j = 1,2 \)). In order to prove that Eq. (22) comes from Eq. (5), we first note that the properties of the function \( f(x) \) of Eq. (13) allow us to write

\[
\bar{\mathcal{O}} = \sqrt{\bar{\mathcal{O}}_1} + \sqrt{\bar{\mathcal{O}}_2} = \sqrt{\frac{a_1 b_1}{a_2 b_2} \bar{\mathcal{O}}_1 + \frac{a_2 b_2}{a_1 b_1} \bar{\mathcal{O}}_2}.
\]

(24)

Then, by applying the Cauchy-Schwarz inequality and using definitions (7) and (23) we can build up the following chain of relations:

\[
|\langle \hat{c}_1 \hat{c}_2 \rangle|^2 = \sum_k w_k |\langle \hat{c}_1 \rangle_k | |\langle \hat{c}_2 \rangle_k |^2 \\
\leq \sum_k w_k |\langle \hat{c}_1 \rangle_k |^2 \sum_k w_k |\langle \hat{c}_2 \rangle_k |^2 \\
= ||\hat{c}_1|| ||\hat{c}_2|| \left[ \sum_k w_k |\langle \hat{c}_1 \rangle_k | \right] \left[ \sum_k w_k |\langle \hat{c}_2 \rangle_k | \right] \\
= ||\hat{c}_1|| ||\hat{c}_2|| \mathcal{O}_1 \mathcal{O}_2
\]

(25)

or

\[
\mathcal{O}_1 \mathcal{O}_2 \geq \frac{|\langle \hat{c}_1 \hat{c}_2 \rangle|^2}{||\hat{c}_1|| ||\hat{c}_2||}.
\]

(26)

Substituting Eqs. (24) and (26) into Eq. (5) we finally get Eq. (22). Equations (5) and (22) give the same separability criterion when \( \hat{c}_j \) is a real number \( c_j \), and the parameters \( a_j, b_j \) satisfy the condition \( a_j b_1 c_1 = \pm a_2 b_2 c_2 \). An example of this situation has been presented in Ref. [11].

Summarizing, we have proved that condition (5) is stronger than the criteria of Refs. [9,11]. This is depicted in Fig. 2, where inequality (5) determines a zone under the solid hyperbola where we can only find entangled states; separable states must lie above this curve. Notice however that entangled states could also lie above the solid hyperbola since condition (5) is only sufficient for entanglement. On the other hand, also the condition (22) determines a portion of the plane where only entangled states can live: that below the dashed hyperbola. However, this part is entirely included in the portion subtended by the solid hyperbola. Finally, criterion (20) determines a straight line inclined at \(-45^\circ\) which, in general, is not tangent to the solid hyperbola representing
condition (5). Also in this case the portion of the plane reserved to an entangled states is included in the portion delimited by the hyperbola of Eq. (5). This shows the generality of the criterion presented in Sec. II.

A couple of interesting connections can be also established when comparing the criterion of Eq. (5) with the weaker Einstein-Podolsky-Rosen (EPR) criterion discussed in Ref. [7] and with the Simon criterion [10]. In fact Eq. (5) and the weaker EPR criterion are essentially equivalent when applied to observables \( \hat{r}_j, \hat{s}_j \) with trivial commutation rules. In order to show this, it is sufficient to observe that the uncertainties \( \langle (\Delta \hat{u})^2 \rangle \) and \( \langle (\Delta \hat{v})^2 \rangle \) give an upper bound for the errors in the inferred measurements of the observables \( a_i \hat{r}_1 \) and \( b_1 \hat{s}_1 \) obtained through a direct measurement of the operators \( -a_2 \hat{r}_2 \) and \( -b_2 \hat{s}_2 \) on \( \hat{\rho} \) (see Ref. [6,7] for more details about the definition of the inferred measurements). The comparison with Simon’s criterion is obtained considering the case in which \( \hat{r}_j, \hat{s}_j \) are linear combinations of the position \( \hat{q}_j \) and momentum \( \hat{p}_j \) operators of the \( j \)th system, i.e.,

\[
\begin{align*}
\hat{r}_1 &= \hat{q}_1 + \frac{a_3}{a_4} \hat{p}_1, \\
\hat{s}_1 &= \hat{p}_1 + \frac{b_3}{b_4} \hat{q}_1
\end{align*}
\]
\[
\begin{align*}
\hat{r}_2 &= \hat{q}_2 + \frac{a_4}{a_2} \hat{p}_2, \\
\hat{s}_2 &= \hat{p}_2 + \frac{b_4}{b_2} \hat{q}_2,
\end{align*}
\]

(27)

where \( a_3, a_4, b_3, b_4 \) are generic real parameters. Since in this case \( [\hat{q}_j, \hat{p}_j] = i \), Eq. (5) becomes

\[
\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle \geq \frac{1}{2} [ |a_1 b_1 - a_3 b_3| + |a_2 b_2 - a_4 b_4| ]^2,
\]

(28)

which has to be compared with the corresponding equation of Ref. [10], i.e.,

\[
\langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle \geq |a_1 b_1 - a_3 b_3| + |a_2 b_2 - a_4 b_4|.
\]

(29)

It is easy to verify that given \( a_j, b_j \) (\( j = 1, \ldots, 4 \)), the product condition of Eq. (28) implies the sum condition of Eq. (29). However, the necessary criterion for separability of Simon requires that Eq. (29) should be verified for all possible values of the coefficients \( a_j, b_j \) (see Eq. (11) of Ref. [10]). In this case, Eqs. (28) and (29) are equivalent since one can reobtain the first from the second using the same convolution trick as already used in deriving Eq. (15) from Eq. (14). In particular this means that Eq. (28), when considered for all possible values of \( a_j, b_j \), provides a criterion for separability which is necessary and sufficient if applied to Gaussian states.

Finally, it is also possible to establish a connection with the criteria used in Refs. [14,15] for discrete variable systems. In particular, assuming \( \alpha = \beta = 1 \) in Eq. (14) and \( a_1 = 1, a_2 = \pm a_1, b_1 = 1, b_2 = \pm b_1 \), in Eq. (16), we get

\[
\langle (\Delta \hat{u})^2 \rangle \langle (\Delta \hat{v})^2 \rangle \geq |\langle \hat{C}_1 \rangle| + |\langle \hat{C}_2 \rangle|.
\]

(30)

For symmetric condition between the two subsystems, the above equation reduces to Eq. (4) of Ref. [15] with \( \hat{r}_j, \hat{s}_j \) being the fluctuations of the Stokes parameters.

IV. CONCLUSION

In this paper we have studied the connections between the separability condition of the initial state of a bipartite system and the uncertainty relation of a couple of nonlocal observables \( \hat{u}, \hat{v} \) of the two subsystems. In the case where \( \hat{u}, \hat{v} \) are linear combinations of generic operators \( \hat{r}_j, \hat{s}_j \) of the two subsystems, we have derived a mathematical constraint, Eq. (5), that has to be satisfied by a separable system. In general, this relation depends on terms that are not measurable, meaning that Eq. (5) cannot be directly used to test experimentally the separability of the system. However, in many cases of experimental relevance, Eq. (5) can be expressed in terms of measurable quantities (see the discussion at the end of Sec. II), providing a very general necessary criterion for separability, i.e., a sufficient criterion for entanglement. Most importantly, Eq. (5) represents a powerful theoretical tool that can be used to derive new measurable criteria [see, for instance, Eqs. (16) and (22)] and to compare them with other already known criteria (e.g., those given in Refs. [6–11,14,15]).