Separability conditions from entropic uncertainty relations

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We derive a collection of separability conditions for bipartite spin 1/2 systems which is based on the entropic version of the uncertainty relations. A comparison with existing criteria is performed.

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Separability is the property that distinguishes classical statistical ensembles from quantum systems. As a matter of fact, the best definition of entanglement is the formal negation of the separability condition [1]. In spite of this clear logical distinction between separable and entangled states, the border-line among these two sets is very difficult to characterize in practice. In the case of bipartite systems, many necessary conditions for separability (i.e., criteria for entanglement) have been proposed [2]. Some of these conditions pertain to general geometrical properties of separable states which are difficult to observe experimentally. The ones that have a better chance to become an operative procedure for testing the presence of entanglement are those criteria which establish boundaries either on the expectation values [3], or on the statistical variances [4] of some observables. These last separability conditions take advantage of the fact that, when specified on non entangled states, the uncertainty relations of a collection of joint observables $\hat{A}, \hat{B}, \ldots$ (i.e., observables which act non trivially in both the Hilbert spaces which constitute the bipartite system) are forced to satisfy lower bounds which are higher than the ones they have to obey when applied to generic states $\rho$ [5]. The reason for this is that, in general, $\hat{A}, \hat{B}, \ldots$ possess only entangled eigenstates: on such states these observables are allowed to achieve the minimum values of their uncertainties. On the other hand, because no common eigenstate of $\hat{A}, \hat{B}, \ldots$ is separable, it is not possible to minimize the uncertainties of all these observables simultaneously on such configurations.

In this paper we propose an approach to the separability problem of a bipartite spin system based on the entropic uncertainty relation in place of the usual Heisenberg-like uncertainty relations. A somehow analogous endeavor has been undertaken in [11] where the standard Bell’s inequalities where rewritten in terms of entropic quantities. Even though for the case of spins 1/2 very efficient separability criteria are known (e.g. [12]), the strategy we propose can be useful to better understand both the geometrical structure of the Hilbert space of the system and the connections between uncertainty relations and entanglement. The material is organized as follows. First we give a brief review of the entropic relations. Then we derive three different new separability conditions for two spin systems. The paper ends with an example in which we test the relations derived and compare them with existing entanglement criteria.

Introduction.— Entropic uncertainty relations has been proposed as an alternative to the standard Heisenberg-like relations, in the case of observables with non trivial C-number commutators [12]. The basic idea of this approach is to replace the statistical variance with the Shannon entropy as an estimator of the uncertainties associated with the measurement process. Consider for instance an observable $\hat{X}$ with $K$ distinct eigenvalues $x_1, \ldots, x_K$ and spectral decomposition

$$\hat{X} = \sum_{k=1}^{K} x_k \rho_k ,$$

with $\rho_k$ the projector in the eigenspace of $\hat{X}$ relative to the eigenvalue $x_k$. Given a state $\rho$ of the system, we define the entropic uncertainty of $\hat{X}$ as

$$H(\rho) \equiv - \sum_{k=1}^{K} P_k \ln P_k$$

where $P_k \equiv \text{Tr}(\hat{X} \rho)$ is the probability of finding the state $\rho$ in the $k$th eigenspace. $H(\rho)$ can be used to estimate the uncertainty of the outcome of a measurement of $\hat{X}$ on the state $\rho$. In fact, if $\rho$ is one of the eigenvectors belonging, say, to the $k_0$th eigenspace, then $P_{k_0} = \delta_{k_0,k_0}$ and $H(\rho)$ nullifies. On the contrary, if $\rho$ is an equally weighted superposition or mixture of all the eigenstates of $\hat{X}$, the measurement result is maximally undetermined and $H(\rho)$ achieves its maximum value $\ln K$. A few remarks are in order. The standard expression $X, H(\rho)$ for the entropic uncertainty of the operator is given by the Shannon entropy with $\rho$ as states belonging to the Hilbert space dimension, otherwise one can show that $H(\rho)$ of Eq. (2) only if $\hat{X}$ is non degenerate (i.e. if $K$ is equal to the Hilbert space dimension). This quantity coincides with $H(\rho)$ only if $\rho$ is an equally weighted superposition of all the eigenstates of $\hat{X}$, the measurement result is maximally undetermined and $H(\rho)$ achieves its maximum value $\ln K$.

In this paper we propose an approach to the separability problem of a bipartite spin system based on the entropic uncertainty relation in place of the usual Heisenberg-like uncertainty relations. A somehow analogous endeavor has been undertaken in [11] where the standard Bell’s inequalities where rewritten in terms of entropic quantities. Even though for the case of spins 1/2 very efficient separability criteria are known (e.g. [12]), the strategy we propose can be useful to better understand both the geometrical structure of the Hilbert space of the system and the connections between uncertainty relations and entanglement. The material is organized as follows. First we give a brief review of the entropic relations. Then we derive three different new separability conditions for two spin systems. The paper ends with an example in which we test the relations derived and compare them with existing entanglement criteria.

$$H(\rho) + H(\rho) \geq -2 \ln \left( \max_{k,j} \| \rho_k \rho_j \| \right) ,$$

(3)
where \( H(\mathcal{Y}, \rho) \) is the entropic uncertainty of \( \hat{Y} \) and where \( ||\mathcal{O}|| = \max_\psi |\langle \psi | \mathcal{O} | \psi \rangle| \) is the norm of the operator \( \mathcal{O} \). In our approach Eq. 3 replaces the standard uncertainty relation which involves the product of the statistical variances of the two operators. These two relations are not completely equivalent, but both predict that when \( X \) and \( Y \) commute, no constraint is imposed on the accuracy with which we can measure them on the same state (see for example 12, 14).

A separable state of a bipartite system composed of subsystems \( a \) and \( b \) (in our case spins 1/2), is any density matrix \( \rho_{\text{sep}} \) which can be expressed as a convex combination of tensor product states, as

\[
\rho_{\text{sep}} = \sum_n \lambda_n |\psi_n\rangle_a \langle \psi_n| \otimes |\phi_n\rangle_b \langle \phi_n| ,
\]

with \( |\psi_n\rangle_a \) and \( |\phi_n\rangle_b \) pure states of the subsystem \( a \) and \( b \) respectively, and \( \lambda_n \geq 0, \sum_n \lambda_n = 1 \). The aim of this paper is to give a class of entropic relations like Eq. 4 that the states \( \rho_{\text{sep}} \) have to satisfy.

Joint product-observables (a).— As a first example, consider the following joint observables

\[
\hat{X} \equiv \sigma_a^{(1)} \otimes \sigma_b^{(1)} , \quad \hat{Y} \equiv \sigma_a^{(2)} \otimes \sigma_b^{(2)} ,
\]

where \( \sigma_i^{(s)} \), for \( i = 1, 2, 3 \) and \( s = a, b \), are the Pauli matrices acting on the \( s \) spin. Because \( \hat{X} \) and \( \hat{Y} \) commute, the right hand side of Eq. 3 vanishes and no lower bound is required to the sum of the entropic uncertainties of these two operators. For example, one can nullify both \( H(X, \rho) \) and \( H(Y, \rho) \) by choosing \( \rho \) to be the singlet state \( |\Psi_\text{sep}\rangle = (|0, 0\rangle - |1, 1\rangle)/\sqrt{2} \). However, if we restrict our analysis to separable states \( \rho_{\text{sep}} \), it is possible to show that the following inequality applies,

\[
H(X, \rho_{\text{sep}}) + H(Y, \rho_{\text{sep}}) \geq \ln 2 .
\]

This relation can be used to test the presence of entanglement in the system: if some state violates it, such a state cannot be separable. In order to prove Eq. 6 it is sufficient to show that the inequality is valid for every pure separable state, i.e., for any tensor product state \( |\Psi_{\text{sep}}\rangle = |\psi\rangle_a \otimes |\phi\rangle_b \). We can then use the convex property of the Shannon entropy 13, 14, 15 to automatically extend the proof to all the other separable states 11. Expand the state \( |\Psi_{\text{sep}}\rangle \) in the eigenstate basis \( \{ |0\rangle_s, |1\rangle_s \} \) of the \( \sigma_s^{(3)} \) matrices so that,

\[
|\psi\rangle_a = \cos \alpha |0\rangle_a + e^{i\delta} \sin \alpha |1\rangle_a , \quad |\phi\rangle_b = \cos \beta |0\rangle_b + e^{i\gamma} \sin \beta |1\rangle_b ,
\]

where \( \alpha, \beta, \delta \) and \( \gamma \) are real parameters. The observable \( \hat{X} \) has the eigenvalues +1 and −1, which are both two time degenerate and have eigenspaces generated by the vectors \( \{ |0, 0\rangle, |1, 1\rangle \} \) and \( \{ |0, 1\rangle, |1, 0\rangle \} \), respectively. Using the parametrization of Eq. 7, the probabilities of finding the state \( |\Psi_{\text{sep}}\rangle \) in these eigenspaces can be then expressed as,

\[
P_{+, 1} = |\cos \alpha \cos \beta|^2 + |\sin \alpha \sin \beta|^2 , \quad P_{-, 1} = 1 - P_{+, 1} .
\]

Consequently the entropic uncertainty of \( \hat{X} \) is

\[
H(X, |\Psi_{\text{sep}}\rangle) = \mathcal{H}(|\cos \alpha \cos \beta|^2 + |\sin \alpha \sin \beta|^2) ,
\]

with \( \mathcal{H}(x) = -x \ln x - (1-x) \ln (1-x) \) the binary entropy function. In the same way we can calculate the entropic uncertainty of the operator \( \hat{Y} \) and show that for \( |\Psi_{\text{sep}}\rangle \) the following relation applies,

\[
H(X, |\Psi_{\text{sep}}\rangle) + H(Y, |\Psi_{\text{sep}}\rangle) = \mathcal{H}(|\cos \alpha \cos \beta|^2 + |\sin \alpha \sin \beta|^2) + \mathcal{H}(1 - \sin \delta \sin \gamma \sin (2\alpha) \sin (2\beta))/2 .
\]

We are interested in the minimum value achievable by this four parameters function. The analysis is simplified by the fact that the binary entropy \( \mathcal{H}(x) \) is a decreasing function of \( |1 - 2x| \). Hence, for any \( \alpha, \beta \in [0, \pi/2] \) the right hand side of Eq. 10 assumes its minimum value for \( \delta, \gamma = \pm \pi/2 \). Assigning thus the values \( \delta \) and \( \gamma \) the right hand side of Eq. 10 can then be shown to have minimum value equal to \( \ln \frac{2}{3} \) (see Fig. 1 concluding the proof. The condition 12 can be generalized to the case where \( \hat{X} \equiv \sigma_a (\hat{n}_a) \otimes \sigma_b (\hat{n}_b) \), \( \hat{Y} \equiv \sigma_a (\hat{n}_a^+) \otimes \sigma_b (\hat{n}_b^+) \), \( \sigma_s \) being the Pauli matrix of the \( s \) spin in the \( \hat{n}_s \) direction, and \( \hat{n}_s \) a unit vector orthogonal to \( \hat{n}_s \).

Joint product-observables (b).— Entropic uncertainty relations can be derived for more than two observables at a time. At least in the case of spins, this produces inequalities which cannot be obtained by averaging over collections of uncertainty relations involving couples of observables 15. In order to exploit this effect, we introduce a third joint observable, \( \hat{Z} \equiv \sigma_a^{(3)} \otimes \sigma_b^{(3)} \) and derive a separability condition which is independent from Eq. 3. On one hand, since \( \hat{Z} \) commutes with the operators \( \hat{X}, \hat{Y} \) of Eq. 3, for a generic state \( \rho \) we have

\[
H(X, \rho) + H(Y, \rho) + H(Z, \rho) \geq 0 ,
\]
with the triplet state $|\Psi_-\rangle$ providing an example of a state that satisfies the identity. On the other hand, when the quantity on the left hand side of Eq. (11) is evaluated on the pure separable state $|\Psi_{\text{sep}}\rangle$, it gives

$$
H(X, |\Psi_{\text{sep}}\rangle) + H(Y, |\Psi_{\text{sep}}\rangle) + H(Z, |\Psi_{\text{sep}}\rangle) = \mathcal{H}(|\cos \alpha \cos \beta|^2 + |\sin \alpha \sin \beta|^2) + \mathcal{H}(1 - \sin \delta \sin \gamma \sin(2\alpha) \sin(2\beta)/2) + \mathcal{H}(1 + \cos \delta \cos \gamma \sin(2\alpha) \sin(2\beta)/2),
$$

which replaces Eq. (10). This expression is quite difficult to study analytically, however a simple numerical analysis shows that its minimum value is equal to $2 \ln 2$ (see Fig. 1). The convexity property of the Shannon entropy then implies that for all separable states $\rho$ the inequality Eq. (11) can be replaced by

$$
H(X, \rho_{\text{sep}}) + H(Y, \rho_{\text{sep}}) + H(Z, \rho_{\text{sep}}) \geq 2 \ln 2.
$$

This relation provides a weaker separability condition, i.e. a more sensitive entanglement criteria, than Eq. (6).

In fact, since each of the operators $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ has only $K = 2$ orthogonal eigenspaces, the quantities $H(X, \rho_{\text{sep}})$, $H(Y, \rho_{\text{sep}})$ and $H(Z, \rho_{\text{sep}})$ are always smaller than $\ln 2$. Using this property it is straightforward to show that Eq. (11) implies Eq. (6). This means that if a state $\rho$ is entangled according to Eq. (6) (i.e. if $\rho$ violates such inequality), then it is also entangled according to Eq. (11) (i.e. $\rho$ violates also this inequality). The opposite, however, is not true: entangled states that satisfy the inequality (6) but not the inequality (11) exist (e.g. see the last paragraphs of this paper).

**Joint sum-observables.**— As a last example of entropic separability conditions, we redefine the observables $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ to be proportional to three orthogonal components of the total spin of the system, namely,

$$
\hat{X} \equiv \sigma_a^{(1)} \otimes 1_b + 1_a \otimes \sigma_b^{(1)}
$$

$$
\hat{Y} \equiv \sigma_a^{(2)} \otimes 1_b + 1_a \otimes \sigma_b^{(2)}
$$

$$
\hat{Z} \equiv \sigma_a^{(3)} \otimes 1_b + 1_a \otimes \sigma_b^{(3)},
$$

where $1_s$ is the identity operator on the spin $s$. Each of these operators has a spectral decomposition (2) with three orthogonal eigenspaces where the total spin of the system, namely,

$$
H(X, \rho_{\text{sep}}) + H(Y, \rho_{\text{sep}}) + H(Z, \rho_{\text{sep}}) \geq 3 \ln 2.
$$

The proof proceeds as the previous ones: we evaluate the left hand side of Eq. (15) on the state $|\Psi_{\text{sep}}\rangle$ and we look for its minimum value. The expression obtained in this case is even more complicated than (10) and (12). Nevertheless, numerical analysis can be used to verify that the minimum is $3 \ln 2$.

**Discussion.**— Equations (6), (13) and (15) are the main results of this paper: they are examples of how one can construct separability conditions starting from entropic uncertainty relations. All of these inequalities derive from the same basic property of the joint observables we have considered. Namely, the fact that the common eigenstates of such operators are entangled. As in the case of separability conditions derived from the Heisenberg-like uncertainty relation, this property can be exploited to impose bounds on the minimum indetermination that can be achieved when measuring simultaneously these operators on separable states. In this final section we analyze the sensitivity in detecting the presence of entanglement of the inequalities (6), (13) and (15), comparing it with the entanglement criteria developed in [8,10]. Following the suggestion of [10] we introduce the set of Werner states

$$
w(p) = \frac{1 - p}{4} 1_a \otimes 1_b + p |\Psi_\downarrow\rangle/\langle \Psi_\downarrow|,
$$

where $|\Psi_\downarrow\rangle$ is the triplet state and $p \in [0,1]$. The density matrices $w(p)$ are separable if and only if $0 \leq p \leq 1/3$: for higher values of the parameter $p$ the states in Eq. (10) are entangled [17].

To test the relations (6) and (13) we need to evaluate the entropic uncertainties defined in Eq. (2) for each one of the operators $\sigma_a^{(i)} \otimes \sigma_b^{(i)}$, with $i = 1, 2, 3$. However, since the states $w(p)$ are rotational invariant, these quantities are identical and it is sufficient to evaluate only one of them. In particular consider $\hat{X} = \sigma_a^{(1)} \otimes \sigma_b^{(1)}$. The projector operators in the eigenspaces of this observable are $X_{+1} = [0,0] \langle 0,0| + [1,1] \langle 1,1|X_{-1} = [0,1] \langle 0,1| + [1,0] \langle 1,0|$, so that the probabilities $P_0$ of Eq. (2) for the state $w(p)$ are $P_{\pm 1} = (1 \pm p)/2$. Consequently, we have

$$
H[\hat{X}, w(p)] + H[\hat{Y}, w(p)] = 2 \mathcal{H} \left( \frac{1 + p}{2} \right),
$$

$$
H[\hat{X}, w(p)] + H[\hat{Y}, w(p)] + H[\hat{Z}, w(p)] = 3 \mathcal{H} \left( \frac{1 + p}{2} \right),
$$

with $\mathcal{H}(x)$ the binary entropy function. According to our analysis we can conclude that the state $w(p)$ is entangled if Eq. (17) or (18) violates the lower bounds established by Eqs. (6), (13), respectively. It can be verified that, in the first case, this happen for $p > .78$, while in the second case it is sufficient to have $p > .65$. [This is in agreement with the fact Eq. (13) is a weaker separability condition than Eq. (6)].

Let us now turn to the sum-observables defined in Eq. (14). Also here, it is sufficient to evaluate only the entropic uncertainty of one of them, say the first one. In this case the eigenspaces projectors are $X_{+2} = [0,0] \langle 0,0|,
\( X_0 \equiv |1, 0\rangle \langle 1, 0 | + |0, 1\rangle \langle 0, 1 | \) and \( X_{-2} \equiv |1, 1\rangle \langle 1, 1 | \), which give the probabilities \( P_{+2} = (1 - p)/4 \) and \( P_0 = (1 + p)/2 \) when applied to the Werner state \( w(p) \). The entropic uncertainty of the observables \( \hat{X}, \hat{Y}, \hat{Z} \) gives thus the value,

\[
H[X, w(p)] + H[Y, w(p)] + H[Z, w(p)] = 3 \left[ F \left( \frac{1 + p}{2} \right) + 2F \left( \frac{1 - p}{4} \right) \right],
\]

with \( F(x) \equiv -x \ln x \). By comparing this function with the separability condition \( \| \hat{X} \|_1 = 1 \), we can establish that \( w(p) \) is entangled for all \( p \) greater than \( .55 \) in fact for such values the function \( F(\ ) \) is smaller than the bound imposed by Eq. \( \| \hat{X} \|_1 = 1 \) to the separable states. The entanglement criterion based on this relation is hence able to recognize more entangled states than the previous two. The reason depends on the fact that the operators (14) provide a decomposition of the two-spins Hilbert space better suited to detect superpositions between the states \( |01\rangle, |10\rangle \) than the decomposition provided by the product operators. Nevertheless, our best example is still not able to pin-point the threshold \( p = 1/3 \) as, instead, the criterion proposed in \( [10] \) does, or even achieve the threshold \( p = 1/2 \) as the criterion proposed in \( [8] \). This limitation however can also turn out to be a positive feature of our criteria. The lacking of sensitivity, in fact, can be interpreted as a more strict requirement imposed by our relations to the states. In this perspective the relations derived here can help to establish a classification of the amount of entanglement in the system.

In conclusion, by analyzing the case of a two-spin system, we have introduced a new class of separability conditions in which the standard Heisenberg relations have been replaced with their entropic counterparts. We have tested these relations on Werner states and shown that the sensitivity of our criteria is lower then the sensitivity of other proposals. Nevertheless the approach we have adopted can be useful in better understanding the geometry of the Hilbert space of composite systems by clarifying the relation between entanglement and uncertainty relations.

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