Decoherence and relaxation of a superconducting quantum bit during measurement

Lin Tian
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Seth Lloyd
Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

T. P. Orlando
Department of Electrical Engineering & Computer Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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During measurement, information is transferred from the measured quantum system to the detector via their coupling. The same coupling that extracts information from the quantum system transmits noise from the detector's environment to the system as well. In this paper, we derive the limit to the measurement efficiency of a superconducting persistent-current qubit measured by a dc superconducting quantum interference device (SQUID) and calculate the noise transmitted to the qubit from the environment of the dc SQUID. The method, with simple linear circuitry correspondence, can be applied to calculate the noise transferred to a quantum system from the environment of an arbitrary external system that interacts with it. The relaxation and decoherence induced by this noise are also estimated.

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I. INTRODUCTION

Quantum computers store and process information on quantum bits. Any coherent controllable two-state quantum system[1] can register a quantum bit; such "qubits" have been realized in a wide variety of physical systems.[2] To collect, manipulate or transfer information from a qubit, we need to entangle the qubit to an external quantum system such as a detector, a radiation source, or another qubit. The external quantum system acts as an information transmission channel that performs an operation on the qubit state and collects information. In addition to the intrinsic decoherence of the qubit, this operation transmits noise from the environment of the external system to the qubit and decoheres the qubit. The stronger the interaction between the qubit and the external system is, the more information is obtained by one measurement and, at the same time, the more noisy the qubit will be. Because the external system typically contains macroscopic, noncoherent elements, it is often exposed to strong environmental noise and becomes a crucial noise source for the qubit. This raises the problem of designing optimized quantum circuits that can maximize the signal-to-noise ratio during information transmission. In this paper we will study the measurement efficiency of the persistent-current qubit measured by a dc superconducting quantum interference device (SQUID) and present a method to derive the noise transferred to the qubit from the detector's environment.

The superconducting persistent-current qubit (pc-qubit) is a solid-state Josephson junction device that stores quantum information on circulating currents.[3] pc-qubits have been successfully fabricated and measured.[4] These measurements are not only the first steps in realizing solid-state quantum computers, but also invoke fundamental studies on verifying quantum mechanics at macroscopic scales.[5,6] In the experiments the SQUID interacts with the qubit and influences the qubit's dynamics. To interpret experimental data correctly, we need to study the interaction between the qubit and the detector carefully to analyze the influence of the detector on the qubit and to extract from the measured data the features that are due to the qubit's behavior.

During a measurement, the detector entangles with the qubit and collects information from the qubit; it then transfers this information into macroscopic distinguishable states that are recorded. Meanwhile, via the same coupling, the detector transmits noise from the output parts of the circuit, which are usually exposed to stronger environmental noise, to the input parts where the qubit is located. The transferred noise will damage the qubit state when it takes a long time to collect the information. This problem exists in many kinds of qubits. In the pc-qubit experiment, the detector is an underdamped dc SQUID, whose critical current is offset by the flux of the two qubit states towards opposite directions. For the offsets to be large enough to resolve the two qubit states, stronger qubit-SQUID coupling is preferred. But noise from SQUID's environment is transmitted to the qubit by the same inductive coupling; moreover, this noise increases with the square of the coupling strength while the offsets increase linearly with the coupling strength. How to design a reasonable circuit to optimize the measurement is thus an important issue.

A similar situation occurs when an ac radiation source is applied to manipulate a qubit. To operate the qubit efficiently, the pass band of the connection circuit between the source and the qubit needs to cover the qubit frequency. This connection lets noise of the same frequency pass as well and may induce strong qubit damping and decoherence. Therefore how to get the best design of an external control source is also an important problem to study.

In this paper we study the pc-qubit measured by a dc SQUID. Both the measurement efficiency and the noise

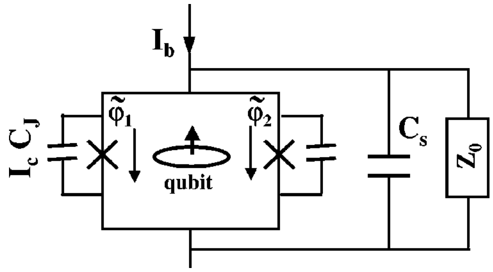

FIG. 1. Persistent-current qubit measured by a dc SQUID. The qubit is in the SQUID loop. I_c is the critical current of the Josephson junctions of the SQUID; C_J is the junction capacitance. $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the gauge-invariant phases of the junctions with their direction indicated by an arrow beside the junction. The SQUID is shunted by a capacitance C_s. Z_0 is the environment of the dc SQUID. The SQUID is biased by ramping current I_b.

transferred to the qubit during the measurement are calculated. The decoherence and relaxation of the qubit are then estimated with the calculated noise spectrum. In Sec. II, we study the interacting system of the qubit and the SQUID in detail and derive the static response of the SQUID to the qubit. Then, we analyze the measurement process for this qubit-SQUID system in Sec. III and derive a limit to the mutual information that could be achieved during one measurement when 1 bit of classical information is encoded in the pc-qubit. In Sec. IV, we apply the Caldeira-Leggett formalism to calculate the spectrum of the noise transferred to the qubit from SQUID's environment by mapping the qubit-SQUID Hamiltonian to a linear circuit. Relaxation and decoherence rates due to this transferred noise are then estimated in terms of the noise spectral density. Conclusions are given in Sec. VII. Note that the study presented here was developed in the course of an ongoing collaboration with the Delft group. In particular, complimentary aspects of the problem were studied by Wilhelm and Grifoni[7] and van der Wal *et al.*[8]

II. QUBIT-SQUID INTERACTION HAMILTONIAN

The superconducting persistent-current qubit[3] is a single superconducting loop that has three Josephson junctions in series. One junction has a slightly different critical current from that of the others. A magnetic flux $f_q\Phi_0$ is applied in the loop, where Φ_0 is the flux quantum. At f_q near 1/2, the lowest two energy levels of this quantum system are nearly localized flux states with opposite circulating currents and are chosen as the qubit states. The two qubit states are analogous to spin states and can be described by $SU(2)$ algebra of the Pauli matrix. By identifying the localized flux states as eigenstates of σ_z, the qubit Hamiltonian is $\mathcal{H}_q=(\epsilon_0/2)\sigma_z+(t_0/2)\sigma_x$, where $\epsilon_0\propto(f_q-1/2)$ is controlled by flux f_q in the qubit loop and t_0 is the tunneling between the two localized flux states. Typical parameters are $\epsilon_0=10$ GHz at $f=0.495$ and $t_0=1$ GHz.

The qubit state can be measured by inductively coupling the qubit to a dc SQUID (Ref. 4) as shown in Fig. 1. The flux of the qubit shifts the flux in the SQUID loop and, hence, the effective critical current of the SQUID. During measurement, a bias current I_b is ramped through the SQUID; the switching current, where the SQUID switches to a finite voltage state, is then measured. Due to quantum fluctuation and thermal activation, the SQUID switches before the critical current and has a finite distribution. The average switching current shifts with the effective critical current and reflects the probability of the two qubit states. In practice, a large capacitance shunts the SQUID to suppress fluctuations and reduce the width of switching current distribution.

When taking into account the self-inductance, the SQUID has two independent variables: the inner variable phase $\tilde{\varphi}_m$ that represents the circulating current of the SQUID loop and the external variable phase $\tilde{\varphi}_p$ that represents a quantum particle in a washboard potential that is tilted by ramping current I_b. The external variable is in a metastable state when the potential is tilted by I_b. A detailed study of the quantization of the SQUID Hamiltonian can be found in Ref. 9. Here we only consider the case when the two junctions are symmetric.

After linearizing the potential energy near the energy minimum, the SQUID variables behave as harmonic oscillators interacting with each other. We have the following approximated Hamiltonian for the qubit-SQUID system:

$$\mathcal{H}_t=\mathcal{H}_q+\frac{\mathrm{P}_m^2}{2m_m}+\frac{1}{2}m_m\omega_m^2(\varphi_m+\delta\varphi_0\sigma_z)^2+\frac{\mathrm{P}_p^2}{2m_p}+\frac{1}{2}m_p\omega_p^2\varphi_p^2+J_1\varphi_m\varphi_p, \quad (1)$$

where $\mathcal{H}_q$ is the qubit Hamiltonian. The phases $\varphi_m=\tilde{\varphi}_m-\tilde{\varphi}_m^0$ and $\varphi_p=\tilde{\varphi}_p-\tilde{\varphi}_p^0$ are the oscillator coordinates relative to the energy minimum $(\tilde{\varphi}_m^0,\tilde{\varphi}_p^0)$. P_m, P_p are the momenta of the inner and the external oscillators, and are conjugate operators of the corresponding phases. The oscillator masses are $m_m=2C_J(\Phi_0/2\pi)^2$ and $m_p=(\Phi_0/2\pi)^2(C_s+2C_J)$, where C_J is the capacitance of the junctions and C_s is the shunt capacitance as shown in Fig. 1. The inner oscillator frequency depends on the self-inductance of the SQUID L_{dc} and the critical current of the junctions I_c as $\omega_m=\sqrt{2/L_{dc}C_J}$. In the experiment, the self-inductance of the SQUID is weak with $\beta_L=2\pi L_{dc}I_c/\Phi_0=0.004$. Hence $\omega_m\approx10^3$ GHz is higher than all the other relevant energies in the qubit-SQUID system. As a result, the inner oscillator is enslaved to the qubit and follows the qubit's dynamics even during qubit operation. The external oscillator frequency depends on the ramping current as $\omega_p=\omega_p^0[1-(I_b/I_c^{eff})^2]^{1/4}$ where $\omega_p^0=\sqrt{2\pi I_c^{eff}/C_s\Phi_0}$ is the oscillator frequency at zero current and I_c^{eff} is the effective critical current of the SQUID under external flux. The inner oscillator offset $\pm\delta\varphi_0\sigma_z$ originates from the inductive interaction between the qubit and SQUID with $\delta\varphi_0=\pi M_qI_{cir}/\Phi_0$, where I_{cir} is the circulating current of the localized states of the qubit and M_q is the mutual inductance. The J_1 term is the bilinear coupling between φ_m and φ_p at the potential energy minimum and is determined by the ramping current I_b. We have J_1

$=|\tan\tilde{\varphi}_m^0|I_b\Phi_0/2\pi$. When the ramping current is turned off, the J_1 coupling disappears, and φ_m and φ_p interact via a higher-order term $\varphi_m\varphi_p^2$ which brings negligible entanglement with the qubit state. Typical numbers for the SQUID are $E_J^{dc}=40$ GHz with $I_c^{dc}=80$ nA, $C_J=2$ fF, $C_s=5$ pF, $L_{dc}=16$ pH, and $M_q=8$ pH. And $\delta\varphi_0\approx 0.002$, $\omega_p^0=1.3$ GHz, and $\omega_p=1.0$ GHz at $I_b=0.8I_c^{eff}$.

This linear model omits the escape of the particle from the washboard potential which becomes stronger as I_b increases. As we mainly use this model to discuss the decoherence and relaxation of the qubit by the SQUID's environment during the ‘‘pre-escape'' entanglement process, this model is valid as far as ω_p is much smaller than the energy barrier (which is true until $I_b\approx 0.95$). We want to point out that the escape from the washboard potential, which is a crucial step in this measurement and will be discussed in the next section, is not included in this model.

When the qubit stays in an eigenstate of σ_z, the response of the SQUID at ramping current I_b can be derived by a perturbation approach. Assuming qubit in $|\uparrow\rangle$ and taking J_1 term as perturbation, the unperturbed eigenstates are $|\uparrow,n_m^\uparrow,n_p\rangle$, where $|n_p\rangle$ are the external oscillator's number states and $|n_m^\uparrow\rangle=e^{\delta\varphi_0\partial/\partial\varphi_m}|n_m\rangle$ are inner oscillator's number states shifted by $\delta\varphi_0$ by interacting with the qubit. The perturbed ground state of the qubit-SQUID system is

$$|\psi_g^\uparrow\rangle=|\uparrow,0_m^\uparrow,0_p\rangle-\frac{J_1\delta\varphi_0 x_p}{\hbar\omega_p}|\uparrow,0_m^\uparrow,1_p\rangle-\frac{J_1 x_m x_p}{\hbar\omega_m}|\uparrow,1_m^\uparrow,1_p\rangle, \tag{2}$$

where $x_m=\sqrt{\hbar/2m_m\omega_m}$ and $x_p=\sqrt{\hbar/2m_p\omega_p}$ are the widths of the ground-state wave function. The state includes contributions from $|1_p\rangle$. Hence, the average of φ_p increases linearly with I_b as $\langle\varphi_p\rangle_0=L_J J_1(2\pi/\Phi_0)^2\delta\varphi_0$, where $L_J=(C_s\omega_p^2)^{-1}$ is the dynamic inductance of the external oscillator. When the ramping current is off, $\varphi_p=0$, the external oscillator responds negligibly to the inner oscillator and is effectively decoupled from the inner oscillator. When the ramping current is on, the external oscillator becomes entangled with the inner oscillator. Given the parameters from the experiment and at $I_b=0.8I_c^{eff}$, $\langle\varphi_p\rangle_0\approx 0.002$.

Compared with the Stern-Gerlach experiment, the qubit–inner-oscillator system acts as the spin of a particle passing through the gradient field. The external oscillator acts as the spatial degree of freedom of this particle. As the particle passes the field, the spatial wave function of the particle becomes separated for the two spin states; by recording the spatial distribution, the probability of different spin states can be obtained. In the SQUID, as the ramping current increases, the states of the external oscillator become separated in its coordinate space in correlation with the two qubit states; by detecting the switching current of the SQUID, the qubit state is detected.

III. LIMITS TO MEASUREMENT OPTIMIZATION

In the above measurement, the same measurement needs to be repeated many times to obtain a satisfactory switching current histogram that can resolve the qubit states. It is thus crucial to analyze the factors that limit the efficiency of the SQUID measurement scheme. This will help to design experiments that can achieve better efficiency. We study the problem from the aspect of extracting classical information encoded in the pc-qubit.

In the experiment, the qubit interacts with the SQUID's inner oscillator via mutual inductance all the time and the flux of the qubit is detected by the inner oscillator even when the measurement is not on, while the switching current histogram is the directly observed physical quantity. We can divide the qubit-SQUID system into two parts: the measured system that includes the qubit and the inner oscillator of the SQUID and the ‘‘meter'' that is the external oscillator of the SQUID. The current ramping process entangles the system with the meter. When the SQUID switches, the meter variable escapes from the supercurrent state to the finite voltage state, and a macroscopically distinguishable record is obtained; in this process, the coherence of the system is completely destroyed by quasiparticle excitations at gap voltage.

The histogram of the switching current is affected by many factors: the critical current of the SQUID, fluctuations and the time dependence of the ramping current, etc. Flux in the SQUID loop changes by $\pm\delta\varphi_0$ due to the qubit states; as a result, the effective critical current I_c^{eff} is shifted by $\Delta I_c=\pm I_c^{eff}\delta\varphi_0|\tan\tilde{\varphi}_m^0|\simeq\pm 10^{-3}I_c^{eff}$, respectively, which results in a shift of the histogram of the same order. Due to strong quantum fluctuation and thermal activation, this shift is much smaller than the width of the histogram,[12] and hence the switching current in any given measurement is not perfectly correlated with the state of the qubit. In other words the measurement is not strictly speaking a von Neumann measurement, but rather a more general positive-operator-valued measurement (POVM).[10]

Encode 1 bit of classical information in the pc-qubit. The density matrix of the qubit is $\rho_q=\frac{1}{2}|\uparrow\rangle\langle\uparrow|+\frac{1}{2}|\downarrow\rangle\langle\downarrow|$ and the Shannon entropy is $H_1=1$. To extract this information from the qubit, we measure the qubit state by ramping a current through the SQUID and measure the switching current histogram. The density matrix of the SQUID is $\rho_{sq}=\frac{1}{2}|\phi_\uparrow\rangle\langle\phi_\uparrow|+\frac{1}{2}|\phi_\downarrow\rangle\langle\phi_\downarrow|$, where $|\phi_s\rangle$ is the SQUID state corresponding to qubit state $|s\rangle$. The probability for the SQUID to switch at current I_b and qubit state $|s\rangle$ is $P(I_b|s)=\mathrm{tr}(\hat{A}_{I_b}|\phi_s\rangle\langle\phi_s|)$, where $\hat{A}_{I_b}$ is the positive operator at I_b. The probability that the SQUID switches at I_b is $P(I_b)=\mathrm{tr}(\hat{A}_{I_b}\rho_{sq})=\frac{1}{2}P(I_b|\uparrow)+\frac{1}{2}P(I_b|\downarrow)$. By the Bayesian theorem, the conditional probabilities for the qubit state when switching occurs at I_b are $P(\uparrow|I_b)=\frac{1}{2}P(I_b|\uparrow)/P(I_b)$ and $P(\downarrow|I_b)=\frac{1}{2}P(I_b|\downarrow)/P(I_b)$. If the histograms of the two qubit states are well separated, given a switching at I_b, one can clearly infer the qubit state from the measurement. But when the histograms of the two qubit states are largely overlapped, which is the case in the current experiment,[4] it is hard to decide the classical bit of information from a single measurement. In terms of the mutual information gained in this measurement, $I_1=H_1-H_{sw}$, where H_1 is the information encoded in the qubit and $H_{sw}=-\Sigma_{I_b}P(I_b)$

$\times[P(\uparrow|I_b)\log P(\uparrow|I_b)+P(\downarrow|I_b)\log P(\downarrow|I_b)]$ is the ensemble-averaged Shannon entropy after the switching event.[11] The mutual information with the experimental parameters[12] is $I_1 \approx 10^{-4}$. This result indicates that in the switching current experiment each measurement only provides limited information and the same measurement has to be repeated many times to decide the qubit states.

Now let us look at what is the best we can achieve with this flux-measuring scheme by coupling the pc-qubit to a SQUID magnetometer. The measurement process is the detection of the inner oscillator states by the external oscillator instead of the direct detection of the qubit state. In other words, the ''meter'' switches according to the inner oscillator states instead of to the qubit state. As a result, with the qubit-SQUID coupling scheme, measurement optimization is limited by how different the two inner oscillator states are corresponding to the two qubit states.

The inner oscillator entangles with the qubit even when the ramping current is not on; furthermore, as $\omega_m \gg \omega_0, \omega_p$, it follows the dynamics of the qubit faithfully even during qubit operation. The inner oscillator is enslaved by the qubit just as an electron is enslaved by the atomic nucleus in a solid. Due to the inductive coupling with the qubit, the inner oscillator states become $|\pm\alpha\rangle = e^{\pm\delta\varphi_0 \partial/\partial\varphi_m}|0_m\rangle$ for the two qubit states respectively, where $|0_m\rangle$ is the oscillator ground state at no interaction and $\alpha = \delta\varphi_0/2x_m$ describes the shift of the oscillator states. Written explicitly, α is determined by the ratio between $\delta\varphi_0 = \pi M_q I_{cir}/\Phi_0$, the coordinate space shift of the oscillator ground state due to coupling with the qubit, and $x_m = \sqrt{\hbar/2M_m\omega_m}$, the width of the ground-state wave function. In the experiment, $\delta\varphi_0 \approx 0.002$ and $x_m \approx 0.1$; we have $\alpha \approx 0.01$, and the overlap of the two coherent states is $\langle -\alpha|\alpha\rangle = e^{-2\alpha^2} = 1-2\times10^{-4}$. Hence, the two states are nonorthogonal and highly overlapped, which means it is hard to distinguish them with any possible measurement. With the qubit encoding 1 bit of classical information, the density matrix of the inner oscillator becomes $\rho_m = \frac{1}{2}[|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|]$. To resolve the information from these two states, a measurement is conducted to resolve the oscillator states. The mutual information in any measurement is limited by the Holevo bound[13]: $I_1 \leqslant S$, where $S = -\mathrm{tr}(\rho_m \ln \rho_m)$ is the von Neumann entropy of ρ_m and is determined by the parameter α. At large α, which indicates well-separated states, the entropy goes to 1; at $\alpha \approx 0.01$, the entropy is very small with $S \approx 0.0015$. This result shows that, with the qubit-SQUID coupling, the most information we could achieve in one measurement is $I_1 \approx 0.0015$. Although this limit is very low, it is one order higher than that of the ongoing experiment.

This analysis indicates that it is possible to improve the measurement efficiency in the flux-measuring scheme while keeping the same qubit and SQUID parameters as in the ongoing experiments. Meanwhile, by adjusting the qubit or the SQUID designs to increase α, better measurement can be expected. Another approach which will be explored in our future work is to go beyond the flux-measuring scheme to exploit the orthogonality of the qubit states, without directly measuring the small flux of the qubit.

IV. DECOHERENCE AND RELAXATION DURING MEASUREMENT

Now we investigate the environment of the qubit-SQUID system. In solid-state systems, decoherence warrants serious attention due to the many redundant degrees of freedom that interact with the qubit. Noise from the direct environment of the pc-qubit was studied in our previous work[14] where the decoherence time can be controlled to be longer than $O(10^{-4})$ sec. However, during the ramping of current I_b, as the qubit and SQUID become entangled, the noise from the environment of the SQUID affects the qubit via their inductive interaction. In this section, we calculate the spectral density of the noise transferred to the qubit from the environment of the SQUID and derive the decoherence and relaxation due to this noise. For simplicity we only discuss the environment of the external oscillator and neglect the environment of the inner oscillator.[15] Although we work on the example of pc-qubit measured by a SQUID, the framework of our discussion is general and can be applied to arbitrary external system interacting with a qubit.

We apply the Caldeira-Leggett approach for the reservoir where the reservoir is modeled as oscillator modes with a continuous spectrum.[16] Localized spin modes[17] can be mapped to oscillator modes when the interaction with the reservoir modes is weak. With the reservoir included, we have

$$\mathcal{H} = \mathcal{H}_t + \sum_\alpha \left[\frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 \left(x_\alpha + \frac{c_\alpha}{m_\alpha \omega_\alpha^2} \varphi_p \right)^2 \right], \quad (3)$$

where $\mathcal{H}_t$ is the Hamiltonian of the qubit-SQUID system; x_α and p_α are the coordinates and momenta of the reservoir modes. The c_α terms are the bilinear interaction between the external oscillator and its reservoir. The direct influence of the reservoir on the associated quantum system—the external oscillator—can be completely described by the spectral density:

$$J_0(\omega) = \frac{\pi}{2\hbar} \sum \frac{c_\alpha^2}{m_\alpha \omega_\alpha} \delta(\omega - \omega_\alpha).$$

The reservoir modes $\{x_\alpha\}$ have no direct interaction with the qubit. They affect the qubit via the interaction between the qubit and SQUID. The qubit sees an effective reservoir which includes both the $\{x_\alpha\}$ modes and a finite number of discrete modes from the external system. In this ''larger'' reservoir, different modes are not independent of each other: namely, modes from the detector interact with the $\{x_\alpha\}$ modes. Meanwhile, the discrete modes of the detector interact with the qubit. In the qubit-SQUID system, the SQUID adds two oscillators to this ''larger'' reservoir. The external oscillator φ_p interacts with the $\{x_\alpha\}$ modes, the inner oscillator φ_m interacts with the external oscillator, and the qubit interacts with this effective reservoir by interacting with φ_m. The noise spectral density for the qubit J_{eff}, according to standard approach,[18–20] can be derived from the dissipation which is the imaginary part of the generalized susceptibility:

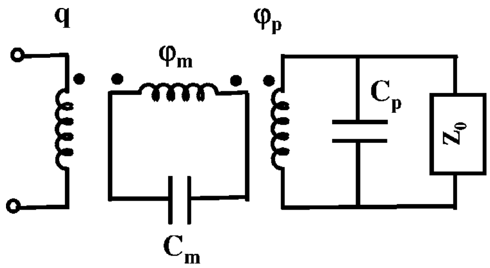

FIG. 2. Equivalent circuit derived from the linearized Hamiltonian of the qubit-SQUID system. The phases q, φ_m, and φ_p are chosen as the independent circuit variables of the three loops of the circuit. The capacitances of the φ_m loop and the φ_p loop are $C_m = 2C_J$ and $C_p = C_s + 2C_J$, respectively. The inductances in the three loops interact via mutual inductances as indicated by the paired dots near the inductances. Z_0 is the environment of the φ_p loop.

$J_{eff}(\omega) = \lim_{\epsilon\to 0}(1/\hbar)\mathrm{Im}[\hat{K}(z)]_{z=\omega - i\epsilon}$, where $\hat{K}(z)$ is the generalized susceptibility of the qubit when taking account of the SQUID.

Instead of calculating the susceptibility $\hat{K}(z)$ directly from the classical equations of motion, the noise can be derived from a simple but general linear circuitry approach which is easily applied to arbitrary external system to derive the transferred noise. Given the Hamiltonian of a linear quantum system, different energy terms can be mapped to linear circuit elements such as inductances, capacitances, and resistors of a linear network. The classical equations of motion for the interacting system are the circuit equations of this linear circuit by Kirchoff's laws. The noise on any quantum variable at zero temperature can hence be calculated from the effective impedance of this circuit:[21] $J_{eff}^{(0)}(\omega) = [\hbar/(2e)^2 \omega \,\mathrm{Re}[Y(\omega)]$ for finite temperature, $J_{eff}^{(T)}(\omega) = J_{eff}^{(0)}(\omega)\coth(\hbar\omega/2k_BT)$, which is the Johnson-Nyquist noise at high temperature.

First, we map the system described by Eqs. (1) and (3) to an equivalent circuit. For each φ_i^2 term we introduce an inductance; for each P_i^2 term we introduce a capacitance in parallel to the inductance; the environment of φ_p is an impedance Z_0 [admittance $Y_0(\omega)$] in parallel to the inductance element φ_p^2. The circuit is shown in Fig. 2. It has three independent flux variables: $\Phi_q = (\Phi_0/2\pi)\langle q\rangle$ $(q = \delta\varphi_0\sigma_z)$, $\Phi_m = (\Phi_0/2\pi)\langle\varphi_m\rangle$, and $\Phi_p = (\Phi_0/2\pi)\langle\varphi_p\rangle$.

From Fig. 2, the currents in the three loops are related to the flux by the inverse inductance matrix:

$$\begin{bmatrix} i_q \\ i_m \\ i_p \end{bmatrix} = \begin{bmatrix} \dfrac{1}{L_q} & \dfrac{4}{L_{dc}} & 0 \\ \dfrac{4}{L_{dc}} & \dfrac{4}{L_{dc}} & \left(\dfrac{2\pi}{\Phi_0}\right)^2 J_1 \\ 0 & \left(\dfrac{2\pi}{\Phi_0}\right)^2 J_1 & \dfrac{1}{L_J} \end{bmatrix} \begin{bmatrix} \Phi_q \\ \Phi_m \\ \Phi_p \end{bmatrix}, \tag{4}$$

where L_q is determined by the self-inductance of the qubit and will not affect our result. Let $v_i, i = q,m,p$ be the voltage of the corresponding inductance. We derive the effective admittance $Y_{eff} = Z_{eff}^{-1} = i_q/v_q$ from circuit equations

$$Y_{eff} = \frac{1}{i\omega L_q} + \frac{16}{\omega^2 L_{dc}^2 Y_m},$$
$$Y_m = i\omega C_m + \frac{4}{i\omega L_{dc}} + \frac{(4\pi^2 J_1)^2}{\omega^2 \Phi_0^4 Y_p}, \tag{5}$$
$$Y_p = i\omega C_p + \frac{1}{i\omega L_J} + \frac{1}{Z_0},$$

where Y_m is the admittance of the circuit without the qubit loop and Y_p is the admittance of the circuit without both the qubit and inner oscillator loops. Plugging Y_{eff} into $J_{eff}^q(\omega)$ and putting the $2\delta\varphi_0$ factor back, the noise spectrum coupling to the qubit is[18–20]

$$J_{eff}(\omega) = (2\delta\varphi_0)^2 \frac{4\hbar}{e^2\omega L_{dc}^2}\mathrm{Re}\left[\left(2i\omega C_J + \frac{4}{i\omega L_{dc}} + \frac{[J_1 4\pi^2]^2}{\omega^2\Phi_0^4\left(i\omega C_s + \dfrac{1}{i\omega L_J} + Y_0(\omega)\right)}\right)^{-1}\right], \tag{6}$$

Note that this linear circuit does not have direct correspondence to the physical system, but comes from the linearized Hamiltonian. This equivalent circuit method is easier to apply to arbitrary external system. Once the linearized Hamiltonian is known, a linear circuit can be obtained whose admittance determines the noise and can be calculated easily.

For the qubit-SQUID system, as $\omega_m \gg \omega_p, \omega_0$, the spectrum at $\omega \ll \omega_m$ can be simplified by ignoring the capacitance C_m term. Assuming an Ohmic environment with resistance R_s and substituting $\delta\varphi_0$ with $\pi M_q I_{cir}/\Phi_0$, we have

$$J_{eff}(\omega) \approx \frac{4(eI_{cir}I_bM_q)^2}{C_s^2\hbar^3 R_s}\frac{\omega}{(\omega^2 - \omega_p^2)^2 + (\omega/R_sC_s)^2}. \tag{7}$$

$J_{eff}(\omega)$ increases with the square of the mutual inductance and the square of the ramping current; hence, when the coupling between qubit and SQUID is stronger, the noise is also stronger. When the ramping current is off, the noise transferred to the qubit is negligible as the entanglement between the inner oscillator and the external oscillator is negligible. For finite ramping current, at low frequency when $\omega \ll \omega_p$, the spectrum increases linearly with ω; compared with the spectrum $J_0(\omega)$, $J_{eff}(\omega)$ is rescaled by a constant factor as $J_{eff} = J_0(\omega)(4\pi\delta\varphi_0 I_b L_J/\Phi_0)^2$. So at low frequency with I_b of the same order of I_c^{eff}, the noise transferred is reduced by an order of $\delta\varphi_0^2$. At high frequency when $\omega \gg \omega_p$, besides the rescaled linear term, another factor shows up as $J_{eff} = J_0(\omega)(4\pi\delta\varphi_0 I_b L_J/\Phi_0)^2(\omega_p/\omega)^4$, and the spectrum decreases with ω^{-3} which further reduces the noise at $\omega = \omega_0$. The SQUID thus acts as a filter that cuts off the high-frequency noise transferred to the qubit. At $\omega \approx \omega_p$, a sharp

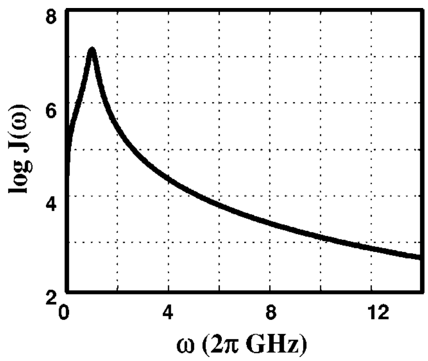

FIG. 3. Effective noise spectrum vs frequency for ramping current $I_b = 0.8 I_c^{eff}$ at zero temperature with an Ohmic environment of $R_s = 100\ \Omega$.

Lorenzian peak appears in the spectrum with a width of $(R_s C_s)^{-1}$. This structure is due to the discrete external oscillator mode of the SQUID that interacts with the reservoir; when the interaction between the SQUID and reservoir goes to zero ($R_s \to \infty$), the peak becomes a δ function. The spectrum is plotted in Fig. 3.

During a measurement, information of the measured system is obtained by entangling this system with a meter variable. In many discussions of measurement, entanglement is accomplished in a very short time during which the evolution of the measured system is neglected; afterward, the meter variable is projected to macroscopically distinguishable states. Hence the dynamics of the measured system has no effect on the result of measurement once an initial state is selected. However, in the measurement of the pc-qubit,[4] entanglement is a slow process (milliseconds) and the interaction between system and meter is much weaker than the qubit energy ω_0. As a result, we have to take into account of the qubit dynamics, including relaxation and decoherence, when studying the measurement. In the following we study the decoherence and relaxation of the qubit due to the effective noise transferred to the qubit from the SQUID's environment.

The transmitted noise affects the qubit via σ_z coupling. As $[\sigma_z, \mathcal{H}_q] \neq 0$, which is generally the case, damping occurs as well as decoherence. We write

$$\sigma_z' = \cos\theta\sigma_z + \sin\theta\sigma_x, \quad \sigma_x' = -\sin\theta\sigma_z + \cos\theta\sigma_x, \tag{8}$$

where θ is the angle between the qubit eigenstate and the σ_z basis with $\cos\theta = \epsilon_0/\omega_0$ and $\sin\theta = t_0/\omega_0$. $\omega_0 = \sqrt{\epsilon_0^2 + t_0^2}$. σ_z' and σ_x' correspond to the Pauli matrices when choosing $|\sigma_z' = \pm 1\rangle$ as the qubit eigenstates. The qubit-reservoir Hamiltonian

$$\mathcal{H}_{q,r} = \frac{\hbar\omega_0}{2}\sigma_z' + \cos\theta\hat{X}_z\sigma_z' - \sin\theta\hat{X}_z\sigma_x', \tag{9}$$

where $\hat{X}_z$ is the coupling between the qubit and the effective noise reservoir whose spectrum is $J_{eff}(\omega)$. In Eq. (9), we have both pure dephasing noise with spectrum $\cos^2\theta J_{eff}(\omega)$ and relaxation coupling with spectrum $\sin^2\theta J_{eff}(\omega)$. Once we know the noise spectrum, the relaxation and decoherence rates can be derived.[22–24] At finite temperature,

$$T_1^{-1} = \frac{t_0^2}{2\omega_0^2} J_{eff}(\omega) \coth\frac{\hbar\omega}{2k_B T}\bigg|_{\omega=\omega_0},$$

$$T_2^{-1} = \frac{\epsilon_0^2}{2\omega_0^2} J_{eff}(\omega) \coth\frac{\hbar\omega}{2k_B T}\bigg|_{\omega\to 0} + \frac{t_0^2}{4\omega_0^2} J_{eff}(\omega) \coth\frac{\hbar\omega}{2k_B T}\bigg|_{\omega=\omega_0}. \tag{10}$$

Due to the reduction factor ω_p^4/ω_0^4 in the spectral density at $\omega = \omega_0$, relaxation is slowed by the filtering of the SQUID. With the system parameters, we calculate the damping time as $\tau_r = 0.15$ sec and the decoherence time as $\tau_d = 2\ \mu$sec at $I_b = 0.8 I_c^{eff}$. The decoherence time is much shorter than the estimated intrinsic decoherence,[14] while relaxation is slow enough that it will not hinder the extraction of qubit information. The noise transferred to the qubit is negligible at $I_b = 0$ when no measurement is being conducted.

Note that the noise increases with the square of the inductive coupling, so are the decoherence and relaxation rates. In contrast, α, the parameter that determines the measurement efficiency, only increases linearly with the inductive coupling. When increasing α by 10 times by adjusting the mutual inductance, the relaxation rate is two orders stronger. This puts an extra restriction on measurement optimization—to keep the noise low for a good enough signal-to-noise ratio.

V. CONCLUSIONS

In this paper we studied the measurement of the pc-qubit by a dc SQUID. We derived the limit to the measurement efficiency in terms of the mutual information. Besides information, the measurement process also transfers additional noise to the qubit from the SQUID's environment. We calculated the noise transferred to the pc-qubit with the Caldeira-Leggett formalism and estimated the relaxation and decoherence of the qubit due to this noise. This study suggests that better readout circuit can be designed to optimize the measurement within the qubit-SQUID coupling scheme.

When calculating the transferred noise, we map the linearized Hamiltonian of the interacting qubit-SQUID system to a linear circuit. By calculating the impedance of this circuit, noise spectrum can be derived directly. This approach can be applied to arbitrary external system, including measurement circuit, control circuit, etc. Note that this linear circuit can not be derived directly from the physical circuit of the interacting systems, but is derived from the linearized Hamiltonian. Several other impedance environments were examined from a direct circuit analysis[8] which is valid in the specific parameter range of the experiment[4] when $\omega_m \gg \omega_0$.

The inductive coupling between the qubit and the SQUID contributes σ_z noise to the qubit. As $[\sigma_z, \mathcal{H}_q] \neq 0$, this noise induces damping as well as decoherence. We found that when the measurement is on, relaxation is reasonably slow and will not prevent the collection of accurate information of the qubit; when the measurement is off, the SQUID introduces negligible noise to the qubit. A more comprehensive study of the structured environment (e.g., the continuous spectrum environment plus the discrete modes of the SQUID) shows that when the qubit frequency is close to the plasma frequency of the external oscillator, the structure of the environment affects the relaxation and decoherence in a sophisticated way.[7]

This study also brings up a general question in quantum information processing: how to divide the system from the environment during any information exchange between the quantum system and an external control devices, such as a detector, a controller, or another qubit. To calculate the noise transferred to the qubit, the SQUID and its environment form an effective environment with spectrum J_{eff}. However, in order to determine the measurement efficiency, the qubit and SQUID's inner oscillator must be treated as a joint quantum system in its own right.

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[15] Similar to the qubit's intrinsic noise, the most significant noise on the inner oscillator comes from nuclear spins and flux fluctuation. As the size of the SQUID loop is of the same order as that of the qubit, the direct noise on the inner oscillator is of the same order as the direct noise on the qubit. When transferred to the qubit, this noise is reduced by a factor of $\delta\phi_0^2(\ll 1)$ and is negligible. The external oscillator is exposed to a much more noisy environment as it interacts directly with external electronics.

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