Optimal Permutation Codes for Uniform Sources†

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Abstract — **Permutation codes (PC's) are vector quantizers whose code vectors are related by permutations and, in one variant, sign changes. Asymptotically, as the vector dimension grows, optimal PC design is identical to optimal entropy-constrained scalar quantizer (ECSQ) design. However, contradicting intuition and previously published assertions, there are finite block length PC's that perform better than the best ones with asymptotically large length; thus, there are PC's whose performances cannot be matched by ECSQ's. Specific counterexamples are created with a memoryless uniform source.**

A fixed-rate vector quantizer represents a random vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ with an element of the codebook $\mathcal{C} = \{\mathbf{y}_i\}_{i=1}^M$, where each codeword $\mathbf{y}_i$ is in $\mathbb{R}^n$. The rate in bits per scalar sample is $R = n^{-1}\log_2 M$. To minimize the per-sample squared error distortion $n^{-1}\|\mathbf{x} - \hat{\mathbf{x}}\|^2$, the optimal encoder maps $\mathbf{x}$ to the nearest element in the codebook: $\hat{\mathbf{x}} = \operatorname{argmin}_{\mathbf{y}\in\mathcal{C}} \|\mathbf{x} - \mathbf{y}\|$. We consider only the case where $\{x_i\}_{i=1}^n$ is an i.i.d. sequence.

Let $n_1, n_2, \ldots, n_K$ be positive integers which sum to n, and let $\mu_1, \mu_2, \ldots, \mu_K$ be K real numbers satisfying $\mu_1 > \mu_2 > \cdots > \mu_K$. A permutation code has $\mathbf{y}_1 = (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2, \ldots, \mu_K, \ldots, \mu_K)$ where each μ_i appears n_i times. The Variant I PC specified by $\{\mu_i\}_{i=1}^K$ and $\{n_i\}_{i=1}^K$ has a codebook consisting of all the distinct permutations of $\mathbf{y}_1$. It has $M = n!/\prod_{i=1}^K n_i!$ codewords. (Variant II PC's have nonnegative μ_i's and additional codewords generated by sign choices; details and results analogous to those presented here appear in [1].) The peculiar structure of a PC codebook makes optimal encoding very easy [2]: Replace the n_1 largest components of $\mathbf{x}$ with μ_1, the next n_2 largest components of $\mathbf{x}$ with μ_2, and so on.

Let π be a permutation that puts $\mathbf{x}$ in decreasing order and let $(\xi_1, \xi_2, \ldots, \xi_n) = \pi(\mathbf{x})$. (The ξ_j's are *order statistics.*) The distortion is minimized with (see [2])

$$\mu_i = n_i^{-1} \sum_{j=n_1+n_2+\cdots+n_{i-1}+1}^{n_1+n_2+\cdots+n_i} E[\xi_j], \quad i = 1, 2, \ldots, K.$$

The challenge in PC design is thus only the selection of the n_i's.

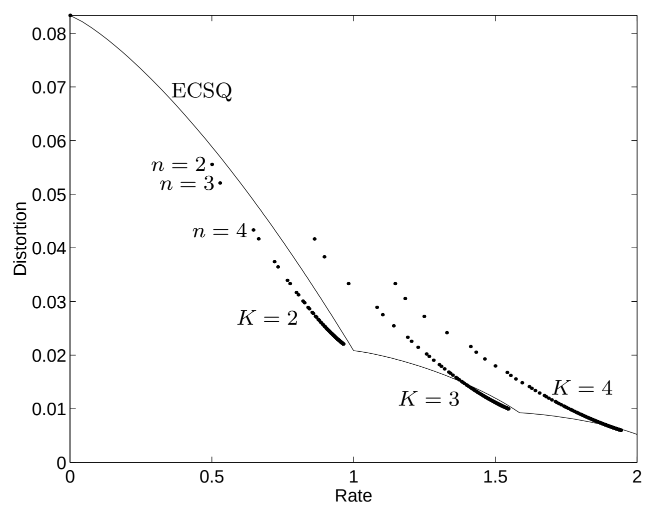
If all the n_i's are large, $n^{-1}\log_2 M \approx -\sum_{i=1}^K \frac{n_i}{n}\log_2\frac{n_i}{n}$. Also, the deterministic quantities $\{E[\xi_j]\}_{j=1}^n$ are distributed identically to a generic source variable x. Thus, PC design becomes identical to entropy-constrained scalar quantizer (ECSQ) design [1, 3]. It is asserted in [3] that "no finite block length PC of fixed rate R can achieve a smaller value of D than that which is approached in the limit as $n \to \infty$, so we conclude that optimum PC's and optimum [ECSQ's] are equivalent in the (R, D) sense." Calculations for a uniform source show that this is false.

Let $\{x_i\}_{i=1}^n$ be uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$. The $E[\xi_j]$'s and optimal μ_i's have simple forms and yield

$$D = \frac{1}{12n(n+1)^2}\left[2n^2 + n + \sum_{i=1}^K {n_i}^3\right].$$

Given a maximum for the rate R, it is difficult to choose n, K and $\{n_i\}_{i=1}^K$ to get a PC with minimum distortion. (Taking n very large is not necessarily good!) However, the convex hull of (R, D) pairs for a given n is limited as follows [1]: $|n_i - n_j| \le 1$ for all i and j.

For any $n \ge 2$, taking $K = 2$ and $(n_1, n_2) = (\lceil n/2\rceil, \lfloor n/2\rfloor)$ gives performance better than ECSQ, as shown in the plot below. With appropriate rounding, the $(n/3, n/3, n/3)$ codes with $n \ge 26$ and $(n/4, n/4, n/4, n/4)$ codes with $n \ge 96$ also beat ECSQ. These operating points also show that the convex hull does not always improve as n is increased. For example, the best convex hull at low rates is obtained with $n = 14$.



References

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