Bounds on the Achievable Region for Certain Multiple Description Coding Problems

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Abstract — An achievable region for the \(L\)-channel multiple description coding problem is presented. This region generalizes previous two-channel results of El Gamal and Cover and of Zhang and Berger. New outer bounds on the rate distortion region for memoryless Gaussian sources with mean-squared error distortion are also derived. For the Gaussian source, the achievable region meets the outer bound at certain points.

I. PROBLEM DESCRIPTION

Consider a source that emits a sequence \(X^N = X^{(1)}; \ldots; X^{(N)}\) of \(N\) independent and identically distributed (i.i.d.) random variables. \(X^N\) is encoded into \(L\) descriptions \(J_1, J_2, \ldots, J_L\) at rates \(R_1, R_2, \ldots, R_L\) nats per source symbol. Suppose that each description is either transmitted error-free or lost completely. Thus the receiver encounters one of \(2^L\) configurations depending on which descriptions are received. Excepting the trivial case where no description is received, we can represent the receiver as a collection of \(2^L - 1\) decoders, where each decoder produces an output based on a non-empty subset of \(\{J_1, \ldots, J_L\}\).

Let \(\mathcal{L} = \{1, \ldots, L\}\) and let \(2^\mathcal{L}\) be its power set. For every \(K \in 2^\mathcal{L}\), let \(X_K^N = X_K^{(1)}; \ldots; X_K^{(N)}\) denote the output of the decoder whose inputs are \(\{J_k : k \in K\}\). Next let \(d_K = E[\frac{1}{N} \sum_{n=1}^{N} \delta_k(X^{(n)}; X_K^{(n)})]\) denote the expected distortion per source symbol associated with the output \(X_K\), where \(\delta_k(\cdot; \cdot)\) is a distortion measure. Our problem is to find the set of rates \(\{R_1, \ldots, R_L\}\) and distortions \(\{d_K : K \in 2^\mathcal{L} - \emptyset\}\) that are achievable in the usual Shannon sense. We call this region the rate-distortion (RD) region.

II. AN ACHIEVABLE REGION

The set difference between collections of sets \(\mathcal{C}\) and \(\mathcal{D}\) is denoted \(\mathcal{C} - \mathcal{D} = \{M \in \mathcal{C} : M \notin \mathcal{D}\}\). Also, we write \(R_K\) as a shorthand for \(\sum_{K \in \mathcal{M}} R_K\) and \(X(\mathcal{C})\) for a collection of random variables \(\{X_N : N \in \mathcal{C}\}\). Our first result is an achievable region for the general \(L\)-description problem.

Theorem 1 Let \(X(\mathcal{L})\) be \(2^L\) finite-alphabet random variables jointly distributed with \(X\). Then the RD region contains the rates and distortions satisfying

\[
\begin{align*}
d_K &\geq E\delta_k(X, X_K) \\
R_K &\geq (|K| - 1)I(X; X_0) - H(X(\mathcal{L})|X) \\
&+ \sum_{M \subset K} H(X_M | X(\mathcal{L} - \{M\}))
\end{align*}
\]

for every \(K \in 2^\mathcal{L} - \emptyset\), where \(|K|\) is the cardinality of \(K\).

In Theorem 1, \(X_0\) is an arbitrary random variable. For \(L = 2\), this result generalizes the result of Zhang and Berger [1]. Additionally, with \(X_0\) set to a constant, e.g. 0, it reduces to the result of El Gamal and Cover [2].

Theorem 1 holds more generally for well-behaved continuous sources if all entropies \(H(\cdot)\) are replaced by differential entropies \(h(\cdot)\). We next focus exclusively on the Gaussian source with mean-squared error distortion.

III. THE QUADRATIC GAUSSIAN CASE

The following outer bound generalizes a result of Ozarow [3]. For this result, a collection of \(M\) disjoint sets \(\{K_m\}_{m=1}^{M}\) is called a partition of a set \(K\) if \(\bigcup_{m=1}^{M} K_m = K\).

Theorem 2 For each \(K \in 2^\mathcal{L}\), the achievable rates \(R_l, \ell \in \mathcal{L}\), and distortions \(d_K, K \in 2^\mathcal{L}\), satisfy

\[
e^{-2R_K} \leq \min_{\{K_m\}_{m=1}^{M}} \inf_{\lambda \geq 0} \left( d_K \left(\frac{\prod_{m=1}^{M} \delta_m + \lambda}{\delta_K + \lambda(1 + \lambda)^{-M}}\right)\right)
\]

where the minimization is performed over all partitions of \(K\).

Example Let the source emit unit-variance i.i.d. Gaussian random variables, and suppose that we care only about the reconstructions from individual descriptions \(X(\{1\}), X(\{2\}), \ldots, X(\{L\})\), and the reconstruction from all the descriptions \(X(\mathcal{L})\). An important operating point is that of equal side distortions and rates on all channels: \(d(\ell) = d\) and \(R_l = R \geq -\frac{1}{2} \log d, \ell \in \mathcal{L}\). For this operating point, we can show using Theorems 1 and 2, that the best central distortion achievable is

\[
d_\mathcal{L} = \sup_{\lambda \geq 0} \left(\frac{e^{-2LR}(1 + \lambda)^{L-1}}{(d + \lambda)^L - e^{-2LR}(1 + \lambda)^{L-1}}\right).
\]

In fact, we can show that the outer and inner bounds meet everywhere on an open set in the vicinity of this point.

REFERENCES