Robust Algorithms for Boundary-Element Integrals on Curved Surfaces

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1 ABSTRACT

This paper presents a new approach to computing $1/r$ singularities on curved panels. By using carefully chosen mapping techniques, a curved panel with curved edges is mapped to a flat panel with straight edges. Analytical formulas for flat panel integrals are then used to give approximate curved panel integration solution. For those curved panels with reasonably large curvatures and smooth edges, this method can efficiently achieve excellent accuracy(10^{-5}% error).

Keywords: BEM, panel integration, singularity, curved panel

2 INTRODUCTION

The geometric complexity of most micromachined devices makes full three-dimensional simulation of an entire device a computationally challenging problem, but much progress has been made over the last decade. Fast boundary-element methods(BEM), based on multipole or precorrected-FFT accelerated iterative algorithms [1,2,3], has made certainly 3-D analyses routine. For example, electrostatic analysis of an entire comb drive can now be performed in minutes[4] rather the days. The enormous computational benefit of accelerated BEM has renewed interest in extending and improving fast BEM methods. In this paper we address a problem common to many BEM methods, that of robustly computing surface integrals of Greens functions with $\frac{1}{r}$ type singularities. The existing procedures either force a piece-wise flat geometric approximation or break down for closely spaced surfaces.

Electrostatically deformed membranes and beams can form pairs of curved structures with a very small gap between the structures. If BEM is used to compute electrostatic forces in these structures, then it is necessary to compute $\int \frac{P}{\|x-x'\|} dS'$ over sections of the curved surface, where the sections are referred to as panels. Here P is a polynomial in x' and the evaluation point x is very close to the surface. Adaptive cubature methods can be applied to this near singular integral [5], but such methods use nested subdivision that can become extremely inefficient as x approaches the surface. Also, the structure can be approximated as piece-wise flat and analytic integration are used[6,7], but then many flat panels are needed to achieve high accuracy. In this paper we introduce an efficient mapping method which can be used to evaluate both the singular and near singular integrals.

3 FLAT PANEL INTEGRATION

In this part, we only consider flat panel integration in the form of $\int \frac{p(x,y,z)}{r^{2n+1}} ds$, $n = 0, 1, \ldots$. Here r is the distance between evaluation points and panel, p is a polynomial of x, y and z. First, a local coordinate system (ξ, η, ζ) is derived so that the panel is put in the $\xi - \eta$ coordinate plane. Major computations are finished in the local coordinate system and solutions are then transferred back to the global system. In order to make computation cheaper, two recurrence schemes are used instead of direct integration. Please refer to [7,8] for details.

4 CURVED PANEL INTEGRATION

In this paper, we consider the example of computing $\int \frac{1}{r} ds$ over a curved panel, though other functions can be integrated in a similar way. To begin, note that any geometrically well defined curved panel with large curvatures and smooth edges can be accurately mapped to flat panel with straight edges though the mapping function may be complicated. We consider this approach because there are formulas for analytically integrating $1/r$ on a flat panel [6,7].

For any curved triangular panel, an obvious reference flat panel is the flat panel defined by the three corners of a curved triangular panel. Then integration over the curved panel may be expressed as:

$$\int_{curve} \frac{1}{r_{ec}} ds_{curve} = \int_{flat} \frac{1}{r_{ef}} \left(\frac{r_{ef}}{r_{ec}} |J| \right) ds_{flat}$$

$$\cong \int_{flat} \frac{P(\xi, \eta)}{r_{ef}} ds_{flat} \tag{1}$$

where $r_{ef} \equiv |X_{eval} - X_{flat}|$ and $r_{ec} \equiv |X_{eval} - X_{curve}|$, X_{eval} is the position of the evaluation point, X_{curve} is the position of a point on the curved panel, X_{flat} is the corresponding point on the flat panel, J is the Jacobian matrix, and $P(\xi, \eta)$ is a polynomial approximate for $\frac{r_{ef}}{r_{ec}}|J|$. If the only singularity is the $\frac{1}{r_{ef}}$ term, it will be easy to find a polynomial $P(\xi,\eta)$ that accurately approximates $\frac{r_{ef}}{r_{ec}}|J|$. In addition, we can then use the formulas for $1/r$ integrals over flat panels. So, the ultimate goal is to find a flat reference panel that makes $\frac{r_{ef}}{r_{ec}}|J|$ as smooth as possible. In Fig.1, a tangent panel to the curved surface is shown. A tangent panel that touches the curved surface at a point closest to the evaluation point on the panel is an ideal choice. For the singular case, the evaluation point is on the surface; for the near-singular case, the evaluation point is very close to the surface. In Fig.2, the definition of the singular and near singular case is shown pictorially. We define the tangent point as the point on the surface closest to the evaluation point. For the tangent panel that passes through the tangent point, the following limits exist when point on the panel approaches the tangent point,

$$\lim_{\text{point}\rightarrow\text{tangent point}} \frac{r_{ef}}{r_{ec}} = 1 \tag{2}$$

Note $|J|$ is smooth function, so the ideal mapping we are looking for is a mapping between curved panel and a tangent panel which coincides at the tangent point. The tangent panel is named the ideal reference flat panel.

The second problem is how to find the polynomial $P(\xi,\eta)$. For curved panel integration, it is suggested that the reference panel be used to set up the local coordinate system. The mapping between curved panel and flat panel can be easily defined as $x = x(\xi,\eta)$, $y = y(\xi,\eta)$, and $z = z(\xi,\eta)$. Then the determinant of Jacobian matrix is

$$|J| = \sqrt{\left\{\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi}\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \xi}\frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi}\frac{\partial z}{\partial \eta}\right)^2\right\}} \tag{3}$$

In the global system, the evaluation point is (X', Y', Z'); the point on the curved panel is $(x(\xi,\eta), y(\xi,\eta), z(\xi,\eta))$. In the local system, the evaluation point is (X, Y, Z); the point on the flat panel is $(\xi,\eta,0)$. $P(\xi,\eta)$ can be analytically expanded and approximated to finite orders if the curved surface has an analytical expression. Here we suggest using a cubature method. This method is to find cubature points (ξ_i,η_i) on the reference panel and its corresponding $\frac{r_{ef}}{r_{ec}}|J|$, then calculate coefficients of certain polynomial by forcing the polynomial to match $\frac{r_{ef}}{r_{ec}}|J|$ at those cubature points. If more cubature points are selected, then a least square method can be used to compute coefficients of the polynomial.

More precisely, assume

$$P(\xi,\eta) = c_{0,0}+c_{1,0}\xi+c_{0,1}\eta+c_{1,1}\xi\eta\ldots\ldots c_{m,n}\xi^m\eta^n \tag{4}$$

Then:

$$\begin{bmatrix} 1 & \xi_1 & \eta_1 & \cdots & \xi_1^m\eta_1^n \\ 1 & \xi_2 & \eta_2 & \cdots & \xi_2^m\eta_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_k & \eta_k & \cdots & \xi_k^m\eta_k^n \end{bmatrix} \begin{bmatrix} c_{0,0} \\ c_{1,0} \\ \vdots \\ c_{m,n} \end{bmatrix} = \begin{bmatrix} P(\xi_1,\eta_1) \\ P(\xi_2,\eta_2) \\ \vdots \\ P(\xi_k,\eta_k) \end{bmatrix} \tag{5}$$

Sometimes it is difficult to find the tangent panel analytically. A short Newton iteration can be used to find approximate tangent panel. In order to guarantee the convergence of Newton iteration, an initial value should be close to the tangent point. Mapping is also an important step; the curved surface must be uniquely mapped to the flat surface. Easily made mistakes are surface overlaps (part of the curved surface can be mapped to two or more flat panels) and holes (part of the curved surface can not be mapped to any flat panels).

5 CURVED PANEL ALGORITHM

The curved panel algorithm can be summarized as

1) Calculate tangent point and tangent flat panel
2) Find a mapping between curved panel and reference panel.
3) Compute cubature points.
4) Curved panel integration
   4.1) Compute $\int_{flat} \frac{1}{r^{2n+1}}ds$, $\int_{flat} \frac{\xi}{r^{2n+1}}ds, \ldots \int_{flat} \frac{\xi^m\eta^n}{r^{2n+1}}ds$ analytically.
   4.2) Find $P(\xi,\eta) = c_{0,0} + c_{1,0}\xi + c_{0,1}\eta + \ldots\ldots + c_{m,n}\xi^m\eta^n$ using (5)
   4.3) Compute $\int_{flat} \frac{P(\xi,\eta)}{r^{2n+1}}ds = c_{0,0}\int_{flat} \frac{1}{r^{2n+1}}ds + c_{1,0}\int_{flat} \frac{\xi}{r^{2n+1}}ds, + c_{0,1}\int_{flat} \frac{\eta}{r^{2n+1}}ds \ldots + c_{m,n}\int_{flat} \frac{\xi^m\eta^n}{r^{2n+1}}ds$

This method is very accurate if the value of $\frac{r_{ef}}{r_{ec}}|J|$ is smooth enough.

A simple curved triangle panel that is part of a sphere is used to test the accuracy of $\int_{curve} \frac{1}{r}ds$. The radius of the panel varies from 2 to 7.5, distance between corners are 1, 1 and $\sqrt{2}$. Note that the larger the radius, the "flatter" the panel. First, the singular case (evaluation point is the centroid of the curved panel) and nearby case (evaluation point is not too close to the surface) are tested. Fig.3 and Fig.4 show that very high accuracy are achieved with low order polynomials $P(\xi,\eta)$.

Second, the near-singular performance of the mapping method is tested. Evaluation points are chosen along the line that connects the center of the sphere and centroid of the flat panel defined by the three corners. Fig.5 shows that the accuracy decreases and then increases. This is caused by the singularity. Fig.6 shows the value of $\frac{r_{ef}}{r_{ec}}$ over the flat panel. The value is one at the tangent point, but a close look at $\frac{r_{ef}}{r_{ec}}$ reveals a peak that can not be easily fitted with a polynomial. Our original mapping is a simple one shown in Fig.7, Fig.6 shows that the value of $\frac{r_{ef}}{r_{ec}}$ drops rapidly and then increases when point moves further away from tangent point. This can not be easily fitted to polynomial. Of course, the accuracy of the mapping method will not be too bad as $\frac{r_{ef}}{r_{ec}}$ is smooth over the panel, but it won't be very good unless a large number of cubature points and high order polynomial are used. In the singular case, $\frac{r_{ef}}{r_{ec}}$ strictly increases or decreases when a point on the panel moves farther away from the singularity and this makes low-order polynomial fitting very accurate. When the evaluation point is far away from the panel, $\frac{r_{ef}}{r_{ec}}$ is also smooth enough to be accurately fitted to a polynomial. The difficulty only happens when evaluation point is not far from the panel and not very close to the panel. Fig.5 shows that using large number of cubature points and high order polynomial can achieve high accuracy but at high cost.

The hat-shape of $\frac{r_{ef}}{r_{ec}}$ at near singular area reflects a problem in the mapping method. Modifying the mapping method can solve the problem. The method used here is to keep the tangent point fixed, and meanwhile scaling (enlarging or shrinking) the reference panel. Fig.7 shows that enlarging the reference panel increase the value of $\frac{r_{ef}}{r_{ec}}$ and makes it smoother. Fig.8 shows the modified mapping can get rid of the hat-shape shown in Fig.6. Fig.9 shows that the scaling modification significantly increases accuracy.

6 SPHERE EXAMPLE

To demonstrate the advantage of using curved panels, consider computing the capacitance of a sphere as the charge distribution is uniform. Table 1 compares the capacitance computed using a flat panel geometric approximation with using curved panels and our new integration method. As is clear from the table, curved panels yields fifty times the accuracy with a tenth of the panels.

Number of Panels	Error
48 Curvilinear Panel	0.0127%
48 Flat Panel	8.659%
768 Flat Panel	0.679%

Table 1: An Example, Capacitance of a Sphere

7 CONCLUSIONS AND ACKNOWLEDGEMENTS

In this paper we described a new mapping method that map curved panels to flat panels, integrations of singular kernels are then computed on the flat panels. We also introduce a cubature method to compute the coefficients of polynomials. Results of the singular, nearby and near-singular integration cases are given to show the efficiency of this powerful method. With excellent accuracy, this method can significantly reduce the number of panels used in discretization.

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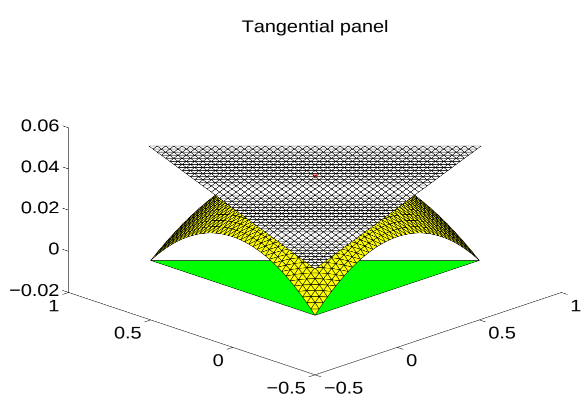

Figure 1: Reference Tangent Panel

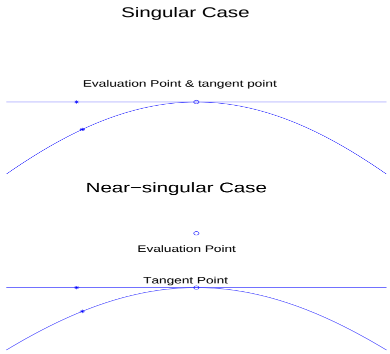

Figure 2: Tangent Point and Evaluation Point

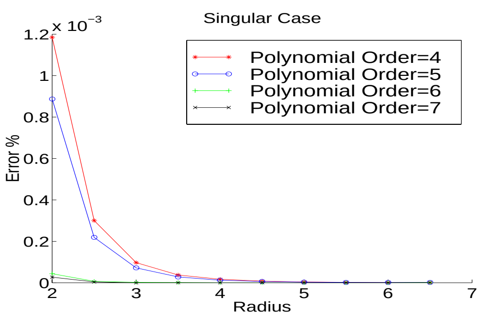

Figure 3: Singular Case Accuracy

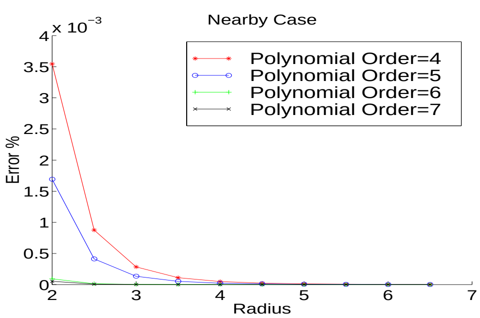

Figure 4: Nearby Case Accuracy

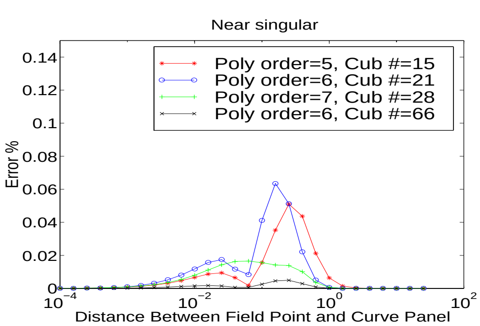

Figure 5: Near Singular Case Accuracy

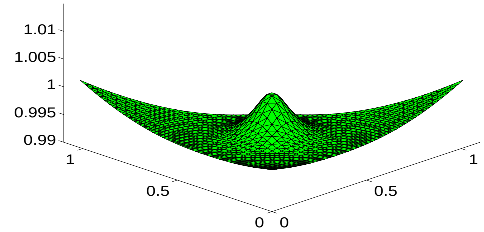

Figure 6: Value of $\frac{r_{ef}}{r_{ec}}$

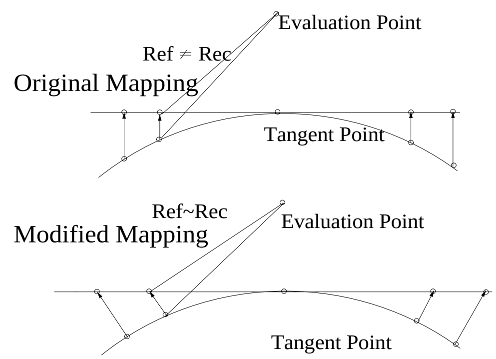

Figure 7: Original Mapping and Modified Mapping

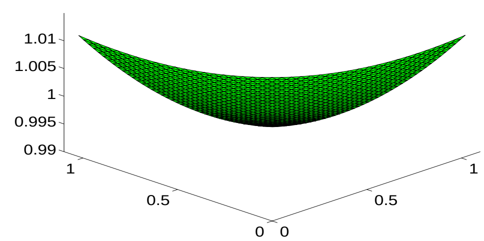

Figure 8: Hat-shape disappears

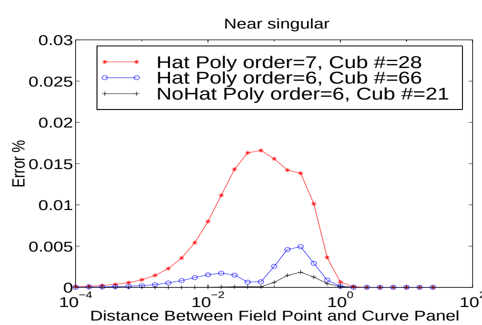

Figure 9: No-hat approach increases accuracy