ADIABATIC ELIMINATION

We derive the four-wave mixing Hamiltonian $\mathcal{H}'$ as the effective Hamiltonian resulting from the adiabatic elimination of excited states in the atom-light Hamiltonian $\mathcal{H}$ in rotating wave approximation. The goal of the adiabatic elimination is to reduce equations of motion for fields (given by the Heisenberg equation) to single equation of motion for the ground state field by setting the time derivatives of the excited state fields to zero. This approximation holds when detuning is much larger than single-photon Rabi frequency as we will see.

The atom-light Hamiltonian $\mathcal{H}$ in the rotating wave approximation is given by

$$\mathcal{H} = \sum_{k,\ell,m} B_{k,\ell,m} \hat{b}^\dagger_\ell \hat{a}_m \hat{c}_k \delta_{\ell-k-m} + \sum_k \frac{\Delta_k}{2} \hat{b}^\dagger_k \hat{b}_k + \text{h.c.}$$

where $\hat{b}^\dagger$ is the creation operator for the excited state atom, $\hat{a}^\dagger$ for the ground state atom, and $\hat{c}^\dagger$ for the pump light photon. The coefficients $B_{k,\ell,m}$ and Δ_k denote the Rabi frequency and detuning from atomic resonance respectively. The symbol "h.c." means Hermitian conjugate of the first two terms. The equations of motion for respective fields are given by (setting $\hbar = 1$)

$$\begin{aligned}
i\dot{\hat{b}}_{\ell'} &= [\hat{b}_{\ell'}, \mathcal{H}] \\
&= \sum_{k,m} B_{k,\ell',m} \hat{a}_m \hat{c}_k \delta_{\ell'-k-m} - \Delta_{\ell'} \hat{b}_{\ell'} \\
i\dot{\hat{a}}^\dagger_{m'} &= [\hat{a}^\dagger_{m'}, \mathcal{H}] \\
&= -\sum_{k,\ell} B_{k,\ell,m'} \hat{b}^\dagger_\ell \hat{c}_k \delta_{\ell-k-m'} \\
i\dot{\hat{c}}^\dagger_{k'} &= [\hat{c}^\dagger_{k'}, \mathcal{H}] \\
&= -\sum_{\ell,m} B_{k',\ell,m} \hat{b}^\dagger_\ell \hat{a}_m \delta_{\ell-k'-m}
\end{aligned}$$

Now we set the first equation of motion to zero. Then we can express $\hat{b}$ in terms of other operators:

$$\hat{b}_{\ell'} = \sum_{k,m} \frac{B_{k,\ell',m}}{\Delta_{\ell'}} \hat{a}_m \hat{c}_k \delta_{\ell'-k-m}$$

We see that in order for this approximation to hold, the condition $B/\Delta << 1$ (small perturbation) is required. Now we feed back the alternate expression for the excited state operator:

$$i\dot{\hat{a}}^\dagger_{m'} = -\sum_{k,\ell} B_{k,\ell,m'} \left(\sum_{k',m''} \frac{B^*_{k',\ell,m''}}{\Delta_\ell} \hat{a}^\dagger_{m''} \hat{c}^\dagger_{k'} \delta_{\ell-k'-m''} \right) \hat{c}_k \delta_{\ell-k-m'}$$

$$i\dot{\hat{c}}^\dagger_{k'} = -\sum_{\ell,m} B_{k',\ell,m} \left(\sum_{k,m'} \frac{B^*_{k,\ell,m'}}{\Delta_\ell} \hat{a}^\dagger_{m'} \hat{c}^\dagger_{k} \delta_{\ell-k-m'} \right) \hat{a}_m \delta_{\ell-k'-m}$$

We write the product of two Kronecker delta functions as single delta function. Then we can re-express the equations of motion as

$$i\dot{\hat{a}}^\dagger_{m'} = -\sum_{k,k',m''} \delta_{k'+m''-k-m'} \left(\sum_{\ell} \frac{B_{k,\ell,m'} B^*_{k',\ell,m''}}{\Delta_\ell} \right) \hat{c}^\dagger_{k'} \hat{c}_k \hat{a}^\dagger_{m''}$$

$$i\dot{\hat{c}}^\dagger_{k'} = -\sum_{k',k,m} \delta_{k+m'-k'-m} \left(\sum_{\ell} \frac{B_{k',\ell,m} B^*_{k,\ell,m'}}{\Delta_\ell} \right) \hat{a}^\dagger_{m'} \hat{a}_m \hat{c}^\dagger_{k}$$

But we see that these equations of motion for (ground state + light) can be derived from the effective Hamiltonian $\mathcal{H}'$:

$$\mathcal{H}' = \sum_{k,\ell,m,n} \delta_{k+\ell-m-n} C_{k,\ell,m,n} \hat{c}^\dagger_k \hat{a}^\dagger_\ell \hat{c}_m \hat{a}_n$$

where

$$C_{k,\ell,m,n} = \sum_p \frac{B_{m,p,n} B^*_{k,p,\ell}}{\Delta_p}$$

i.e. the sum over p is the sum over intermediate states as typically seen in perturbation theory for energy shift.