8.421: Module 3 Problem Set

Due: April 13, 2020

The spontaneous emission rate for a two-level atom in an electromagnetic field is given by

$$\Gamma_{ab} = \frac{4\pi^2}{\hbar V} \left|\hat{\epsilon} \cdot \vec{d}_{ba}\right|^2 \omega_0 n(\omega_0)$$

where the field has frequency ω_0 and polarization $\hat{\epsilon}$ and is confined to a volume V. The number of photons in a frequency interval $d\omega$ is $n(\omega)d\omega$, and $\vec{d}_{ba}$ is the dipole matrix element for the ba transition.

We want to study the spontaneous emission of an atom in a two-dimensional electromagnetic field. We do this by assuming that the atom is in a parallel plate capacitor with plate separation $d \ll \lambda$, the resonant wavelength. Define the direction of the plate separation to be $\hat{z}$.

Problem 1

What are the relevant modes of the electromagnetic field, and what is their mode density?

Solution

Any node of the electric field along z would require $\omega \geq \pi/dc \gg \omega_0$, the electric field is a function only of x and y. Then since the boundary conditions require that transverse polarizations are zero at the two plates, we can write $\vec{E} = f(x, y)\hat{z}$.

Then, the number of modes in a $dk_x \times dk_y$ region is

$$dN = dk_x \cdot dk_y \cdot \frac{A}{(2\pi)^2} = \frac{A}{(2\pi)^2} \frac{\omega \cdot d\omega}{c^2} d\theta$$

where $Ad = V$. The density of modes does not depend on θ, so we can say that $\frac{dN}{d\omega} = \frac{A}{2\pi} \frac{\omega}{c^2}$.

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Problem 2

For a simple transition with only one excited state and one ground state, what is the spontaneous emission rate in terms of $\vec{d}_{ba}$? Compare Γ^{2D} to the 3D decay rate. Is it larger or smaller, and why?

Solution

Following the lecture notes and plugging the above results, we find that

$$\Gamma^{2D} = \frac{4\pi^2}{\hbar V}\omega_0 \left|\hat{\epsilon} \cdot \vec{d}_{ba}\right|^2 \frac{dN}{d\omega}\bigg|_{\omega_0} = \frac{4\pi^2}{\hbar A d}\omega_0 \left|\hat{z} \cdot \vec{d}_{ba}\right|^2 \frac{A}{2\pi}\frac{\omega_0}{c^2} = \frac{2\pi\omega_0^2}{\hbar c^2 d}\left|\hat{z} \cdot \vec{d}_{ba}\right|^2.$$

We already know that

$$\Gamma^{3D} = \frac{4}{3}\frac{\omega_0^3}{\hbar c^3}\left|\vec{d}_{ba}\right|^2.$$

If $\vec{d}_{ba} \parallel \hat{z}$, then

$$\Gamma^{2D} = \frac{3}{4}\frac{\lambda}{d}\Gamma^{3D} \gg \Gamma^{3D}.$$

For $\vec{d}_{ba} \parallel \hat{x}$ or $\hat{y}$, $\Gamma^{2D} \equiv 0$.

Either there is no mode for the atom to decay into, and the lifetime is infinite, or it is much shorter than the 3D lifetime because of the smaller mode volume and therefore higher electric field.

For most atomic transitions, with wavelengths in the hundreds of nm, it is generally hard to reach the 2D regime and see these effects. On the other hand, Rydberg atoms have transitions have wavelengths on the order of a centimeter, so making appropriately sized cavities is possible. This effect was demonstrated in R.G. Hulet, E.S. Hilfer, and D. Kleppner, Phys. Rev. Lett. **55**, 2137 (1985).

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Problem 3

Now find the excited state lifetimes for $p \to s$ and $s \to p$ transitions. How do your answers change when you change the quantization axis of the atoms from $\hat{z}$ to $\hat{x}$? In an experiment, this can be done by changing the direction of an external magnetic field. Write your answers in terms of Γ^{2D}, your answer from the previous problem.

Solution

Start with a quantization axis parallel to $\hat{z}$. For transitions that change the angular momentum projection by ± 1, $\vec{d}_{ba} = (\hat{x} \pm i\hat{y})/\sqrt{2}$, and for transitions that leave the angular momentum unchanged, $\vec{d}_{ba} = \hat{z}$. In the $p \to s$ transition, there are three starting states. For $m = \pm 1$, $\left|\hat{z} \cdot \vec{d}_{ba}\right|^2 = 0$ and for $m = 0$, $\left|\hat{z} \cdot \vec{d}_{ba}\right|^2 = \left|\vec{d}_{ba}\right|^2$. As per the previous solution, we then find $\Gamma_{\pm 1} = 0$ and $\Gamma_0 = \Gamma^{2D}$. For the $s \to p$ transition, in 3D, there are three possible transitions, so the rate for each transition is 1/3 of the spontaneous emission rate. In 2D, only the π transition, from 0 to 0, is allowed, and so it has a decay rate of $\Gamma^{2D}/3$.

Look now at a quantization axis parallel to $\hat{x}$. Now, transitions that do not change the angular momentum projection have $\left|\hat{z} \cdot \vec{d}_{ba}\right|^2 = 0$, while transitions that change angular momentum have $\left|\hat{z} \cdot \vec{d}_{ba}\right|^2 = \left|\vec{d}_{ba}\right|^2/2$. Then, for the $p \to s$ transition, the $m = \pm 1$ states have emission rates $\Gamma^{2D}/2$ and the $m = 0$ state has emission rate 0. For the $s \to p$ transition, the π transition is forbidden, and the $m = \pm 1$ transitions each have emission rates $\Gamma^{2D}/6$.

By changing the quantization axis, we can change which excited states are protected by the 2D density of states.

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Problem 4

Suppose the atom has a $J = 2 \to J' = 1$ transition with rates labeled as follows:

a: $m = 2 \to m' = 1$
b: $m = 1 \to m' = 1$
c: $m = 0 \to m' = 1$
d: $m = 1 \to m' = 0$
e: $m = 0 \to m' = 0$

Find each of these rates in 2D, first in the case where the quantization axis is along $\hat{z}$, and second where the quantization axis is along $\hat{x}$. Write your answers in terms of Γ^{2D}. When excited by light with a particular polarization, a closed (aka cycling) transition will always return to it's initial state. What are the closed transitions in each case?

Solution

From M3.PS1, we know that the squared Clebsch-Gordan coefficients are

a: 1
b: 1/2
c: 1/6
d: 1/2
e: 2/3

Let $\hat{d}$ be the direction of $\vec{d}_{ba}$ for a given transition. The rate of that transition is then $\left|\hat{\epsilon} \cdot \hat{d}\right|^2$ times the squared Clebsch-Gordan coefficient times the spontaneous emission rate.

First look at the quantization axis along $\hat{z}$. Then for transitions that change m by 0, $\hat{d} = \hat{z}$, and for transitions that change m by ± 1, $\hat{d} = (\hat{x} \pm i\hat{y})/\sqrt{2}$. Recalling that 2D restricts us to $\hat{\epsilon} = \hat{z}$, we get the following rates

a: 0
b: $\Gamma^{2D}/2$
c: 0
d: 0
e: $2\Gamma^{2D}/3$

The closed transitions are $m_J = 0 \leftrightarrow m_{J'} = 0$ and $m_J = \pm 1 \leftrightarrow m_{J'} = \pm 1$.

Now look at the case of a quantization axis along $\hat{x}$. In that case, transitions that change m by 0 have $\hat{d} = \hat{x}$, and transitions that change m by ± 1 have $\hat{d} = (\hat{y} \pm i\hat{z})/\sqrt{2}$. This gives us the following rates:

a: $\Gamma^{2D}/2$
b: 0
c: $\Gamma^{2D}/12$
d: $\Gamma^{2D}/4$
e: 0

The closed transitions are $m_J = \pm 2 \leftrightarrow m_{J'} = \pm 1$ and $m_J = \pm 1 \leftrightarrow m_{J'} = 0$.

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Problem 5

For the quantization axis along $\hat{z}$ and along $\hat{x}$, what is the decay rate of an unpolarized sample in the $J = 2$ excited state? Explain your result.

Solution

For the quantization axis along $\hat{z}$, the decay rate is

$$\frac{0 + 2 \times 1/2 + 0 + 0 + 2/3}{5}\Gamma^{2D} = \frac{\Gamma^{2D}}{3}.$$

For the quantization axis along $\hat{x}$, we get

$$\frac{2 \times 1/2 + 0 + 2 \times 1/12 + 2 \times 1/4 + 0}{5}\Gamma^{2D} = \frac{\Gamma^{2D}}{3}.$$

We get the same rate for both choices of quantization axis, because the quantization axis is non-physical - it merely determines how we label the states.

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