Solutions to the module 2

1. **The AC Stark effect and Rayleigh Scattering**

a) The wavefunction of the two-level atom can be written in the form

$$|\psi(t)\rangle = a_g(t)|g\rangle + a_e(t)e^{-i\omega_0 t}|e\rangle \tag{1}$$

The dynamics of $a_e(t)$ is described by the time-dependent Schrodinger equation:

$$i\hbar\dot{a}_e(t)e^{-i\omega_0 t} = -\frac{1}{2}\langle e|D_z|g\rangle\,\epsilon e^{i\omega t} - \frac{1}{2}\langle e|D_z|g\rangle\,\epsilon e^{-i\omega t} \tag{2}$$

then

$$a_e(t) = \frac{\epsilon}{2\hbar}\langle e|D_z|g\rangle\left(\frac{e^{i(\omega_0-\omega)t}}{\omega_0-\omega} + \frac{e^{i(\omega_0+\omega)t}}{\omega_0+\omega}\right) \tag{3}$$

The time-independent terms from the lower limit of integration are omitted. When the ac field is ramped up adiabatically, these terms disappear (check the wiki notes).

The atomic dipole moment of the atom is given by

$$\begin{aligned}\mathbf{d}(t) &= \frac{1}{\hbar}|\langle e|D_z|g\rangle|^2\left(\frac{1}{\omega_0-\omega} + \frac{1}{\omega_0+\omega}\right)\epsilon(t) & (4)\\ &= \alpha(\omega)\,\epsilon(t). & (5)\end{aligned}$$

The quantity $\omega_R = \epsilon\frac{\langle e|D_z|g\rangle}{\hbar}$ is the Rabi frequency. Then the AC Stark effect is given by

$$U_{AC} = -\frac{1}{2}\alpha(\omega)\epsilon^2\overline{\cos^2\omega t} = -\frac{\hbar\omega_R^2}{4}\left(\frac{1}{\omega_0-\omega} + \frac{1}{\omega_0+\omega}\right) \tag{6}$$

One can see that when the drive field is close to the resonance $\omega \approx \omega_0$ the AC Stark effect mostly comes from co-rotating term and the counter-rotating term can be neglected, whereas for the off-resonant light both terms contribute significantly.

These results can be rewritten in another form if we recall that the oscillator strength is unity. We can obtain $|\langle e|D_z|g\rangle|^2 = \frac{\hbar e^2}{2m\omega_0}$. Then the polarizability an the AC Stark shift can be rewritten as:

$$\alpha(\omega) = \frac{e^2}{m}\frac{1}{\omega_0^2-\omega^2}, \qquad U_{AC} = \frac{1}{4}\frac{e^2}{m}\frac{1}{\omega_0^2-\omega^2}\epsilon^2 \tag{7}$$

b)

Using the above results, one finds the likelihood of finding the atom in the excited state in the perturbative approach is

$$\begin{aligned}P_e &= |a_e^2| & (8)\\ &= \frac{\omega_R^2}{4}\left(\frac{1}{(\omega_0-\omega)^2} + \frac{1}{(\omega_0+\omega)^2} + \cdots\right) & (9)\end{aligned}$$

where the $\cdots$ refer to rapidly oscillating terms that we neglect. Thus, the population of the excited state is a sum of squares of co- and counter-rotating contributions.

$$P_e = \frac{\omega_R^2}{4}\left(\frac{1}{(\omega_0-\omega)^2} + \frac{1}{(\omega_0+\omega)^2}\right) = \frac{\epsilon^2 e^2}{8m\hbar\omega_0}\left(\frac{1}{(\omega_0-\omega)^2} + \frac{1}{(\omega_0+\omega)^2}\right) \tag{10}$$

c)

i. In the rotating wave approximation (RWA), when the drive frequency is close to resonance $\omega \approx \omega_0$, we neglect with counter-rotating term. Under this approximation the induced dipole moment of the atom is

$$d_z(t) = \alpha(\omega)\epsilon(t) = \frac{e^2}{2m\omega_0}\frac{1}{\omega_0 - \omega}\epsilon\cos(\omega t) \tag{11}$$

We see that the dipole moment oscillates with at the drive frequency with the amplitude

$$d(RWA) = \frac{e^2}{2m\omega_0}\frac{1}{\omega_0 - \omega}\epsilon \tag{12}$$

The population of the excited state in RWA is

$$P_e(RWA) = \frac{\epsilon^2 e^2}{8m\hbar\omega_0}\frac{1}{(\omega_0 - \omega)^2} \tag{13}$$

ii. Rayleigh scattering can be calculated several ways. Let us use results of classical electromagnetism (for a fully quantum-mechanical derivation, look up Exercize 3 in Cohen-Tannoudji's *Atom-Photon Interactions*). The power emitted by an oscillating dipole of moment $\mathbf{d}(t) = \mathbf{d}\cos\omega t$ is [Jackson, Chapter 9.2]

$$P = \frac{ck^4}{3}|\mathbf{d}|^2. \tag{14}$$

Since each photon carries $\hbar\omega$ of energy, the photon scattering rate is

$$\Gamma_{scat} = \frac{P}{\hbar\omega} = \frac{\omega^3}{3\hbar c^3}|\mathbf{d}|^2. \tag{15}$$

Substituting the amplitude of the dipole moment derived in RWA we obtain

$$\boxed{\Gamma_{scat}(RWA) = \frac{\omega^3}{3\hbar c^3} \times \left(\frac{e^2}{m}\frac{\epsilon}{2\omega_0}\frac{1}{\omega_0 - \omega}\right)^2.} \tag{16}$$

This expression can be regrouped and written in the form

$$\Gamma_{scat}(RWA) = \left(\frac{\epsilon^2 e^2}{8m\hbar\omega_0}\frac{1}{(\omega_0 - \omega)^2}\right) \times \left(\frac{4}{3}\frac{\omega_0^3}{c^3\hbar}\frac{\hbar e^2}{2m\omega_0}\right) \times \left(\frac{\omega}{\omega_0}\right)^3 \tag{17}$$

$$= P_e \times \Gamma \times \left(\frac{\omega}{\omega_0}\right)^3. \tag{18}$$

The Rayleigh scattering rate is given by the probability of being in the excited state, times the spontaneous emission rate from that state, corrected by a phase-space factor which guarantees that an atom in a DC field does not radiate. We have incidentally "determined" the rate of the spontaneous emission

$$\Gamma = \frac{4}{3}\frac{\omega_0^3}{c^3\hbar}\frac{\hbar e^2}{2m\omega_0}$$

Also, as one can see, when $|\omega - \omega_0| \ll \omega_0$, the scattered power is $\hbar\omega\Gamma_{scat}$ proportional to ω^4 – the blue-sky formula.

iii. When the drive frequency ω is far from resonance with the energy separation ω_0 in two-level atom, both co- and counter-rotating terms should be taken into account. We need to use both terms in the equation (4) to employ the same classical formula (15). Thus the Rayleigh scattering rate becomes

$$\boxed{\Gamma_{scat} = \frac{\omega^3}{3\hbar c^3} \times \left(\frac{e^2}{m}\frac{\epsilon}{2\omega_0}\right)^2 \left(\frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega}\right)^2.} \tag{19}$$

One can see that it is no longer proportional to the excited state fraction $P_e = \frac{\epsilon^2 e^2}{8m\hbar\omega_0}\left(\frac{1}{(\omega_0-\omega)^2} + \frac{1}{(\omega_0+\omega)^2}\right)$ and one can regard Rayleigh scattering as spontaneous emission from the excited state only in near-resonant case.

Answer to the optional question:

For the probabilities, one has to add the squares of the amplitude for the co- and counter-rotating terms. For the oscillating dipole moment squared (and therefore the radiated power), one has to square the sum of the two amplitudes, i.e. there is an additional interference term. For low frequency lasers (i.e. CO_2 laser), when the two terms are almost equal, the inclusion of the counter-rotating term doubles the polarizability and the excited state fraction, but quadruples the scattered intensity.

2. **Magic Wavelength Optical Trap**

a) *i.* The AC Stark shift for the S state, as you solved in problem 1, is

$$U_S = \frac{\hbar\omega_{\mathrm{R}}^2}{4}\left(\frac{1}{\omega - \omega_{PS}} - \frac{1}{\omega + \omega_{PS}}\right) \tag{20}$$

$$= \frac{Id_{PS}^2}{2\epsilon_0\hbar c}\left(\frac{1}{\omega - \omega_{PS}} - \frac{1}{\omega + \omega_{PS}}\right), \tag{21}$$

where $\hbar\omega_{\mathrm{R}} = d_{PS}E$, and $I = c\epsilon_0 E^2/2$, where E is the electric field.

ii. Looking at the derivation for the AC Stark shift for the P state, you can see that

$$U_P = -U_S. \tag{22}$$

iii. The relevant parameter is given as

$$E_P^{\text{laser}} - E_S^{\text{laser}} - (E_P^{\text{nolaser}} - E_S^{\text{nolaser}}) = 2U_P. \tag{23}$$

This cannot be zero unless $I = 0$, in which case there will be no trapping laser.

b) *i.* You just need to add the contributions from coupling to both the S and D states, so

$$U_P = -\frac{I}{2\epsilon_0\hbar c}\left[d_{PS}^2\left(\frac{1}{\omega - \omega_{PS}} - \frac{1}{\omega + \omega_{PS}}\right) - d_{DP}^2\left(\frac{1}{\omega - \omega_{DP}} - \frac{1}{\omega + \omega_{DP}}\right)\right] \tag{24}$$

$$= U_{PS} + U_{PD}, \tag{25}$$

where U_{PS} and U_{PD} should be clear.

ii. We need the AC Stark shifts of both the P and S states to be the same, so

$$U_P = U_S \Rightarrow U_{PS} + U_{PD} = U_S \Rightarrow -U_S + U_{PD} = U_S. \tag{26}$$

Solving this for ω gives us

$$\omega = \sqrt{\frac{2n^2 f - f^2}{2n^2 f - 1}}\omega_{PD}. \tag{27}$$

3. **Species-Dependent and Spin-Dependent AC Stark Shift**

a) *i.* I choose to look at the $|S_{1/2}, -1/2\rangle$ state. The answer is the same for the $|S_{1/2}, +1/2\rangle$ by symmetry. The relevant transitions to consider are $|S_{1/2}, -1/2\rangle \to |P_{1/2}, +1/2\rangle$, $|S_{1/2}, -1/2\rangle \to |P_{3/2}, +1/2\rangle$, and $|S_{1/2}, -1/2\rangle \to |P_{3/2}, -3/2\rangle$. Up to an overall constant C, the AC Stark shift is given by

$$U_S = C\left[\frac{2}{3}d_1^2\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega+\omega_1}\right) + \frac{1}{3}d_2^2\left(\frac{1}{\omega-\omega_2} - \frac{1}{\omega+\omega_2}\right) + d_2^2\left(\frac{1}{\omega-\omega_2} - \frac{1}{\omega+\omega_2}\right)\right], \tag{28}$$

where the transitions are in the order mentioned above. Setting this equal to zero, we get

$$\omega^2 = \frac{2d_2^2\omega_1 + d_1^2\omega_2}{2d_2^2\omega_2 + d_1^2\omega_1}\omega_1\omega_2. \tag{29}$$

Plugging in the numbers given, we get

$$\lambda = \frac{2\pi c}{\omega} = 792\text{nm} \tag{30}$$

ii. 792 nm does not agree with the 790 nm laser used in the paper. This discrepancy can be accounted for by taking into account the degeneracy of the fine structure states due to hyperfine levels. This is left as an excersise for the reader.

b) *i.* For the $|S_{1/2}, m_J = +1/2\rangle$ state, the coupling that needs to be considered is $|S_{1/2}, +1/2\rangle \to |P_{3/2}, +3/2\rangle$. This gives us

$$U_{+1/2} = \frac{I}{2\epsilon_0\hbar c}d_2^2\left(\frac{1}{\omega-\omega_2} - \frac{1}{\omega+\omega_2}\right). \tag{31}$$

For the $|S_{1/2}, m_J = -1/2\rangle$ state, the coupling that needs to be considered is $|S_{1/2}, -1/2\rangle \to |P_{3/2}, +1/2\rangle$. This gives us

$$U_{-1/2} = \frac{1}{3}\frac{I}{2\epsilon_0\hbar c}d_2^2\left(\frac{1}{\omega-\omega_2} - \frac{1}{\omega+\omega_2}\right). \tag{32}$$

ii. We can rewrite what we found in *i* as

$$U_{m_J} = \frac{2}{3}\frac{I}{2\epsilon_0\hbar c}d_2^2\frac{2\omega_2}{\omega^2-\omega_2^2}(1+m_J). \tag{33}$$

The second term can be rewritten in the form $g_J m_J \mu_B B_{\text{eff}}$. By doing this we find

$$B_{\text{eff}} = \frac{1}{3}\frac{I}{2\epsilon_0\hbar c}\frac{d_2^2}{\mu_B}\frac{2\omega_2}{\omega^2-\omega_2^2}. \tag{34}$$

4. **Angular Momenta, Magnetic Moments, and g Factors**

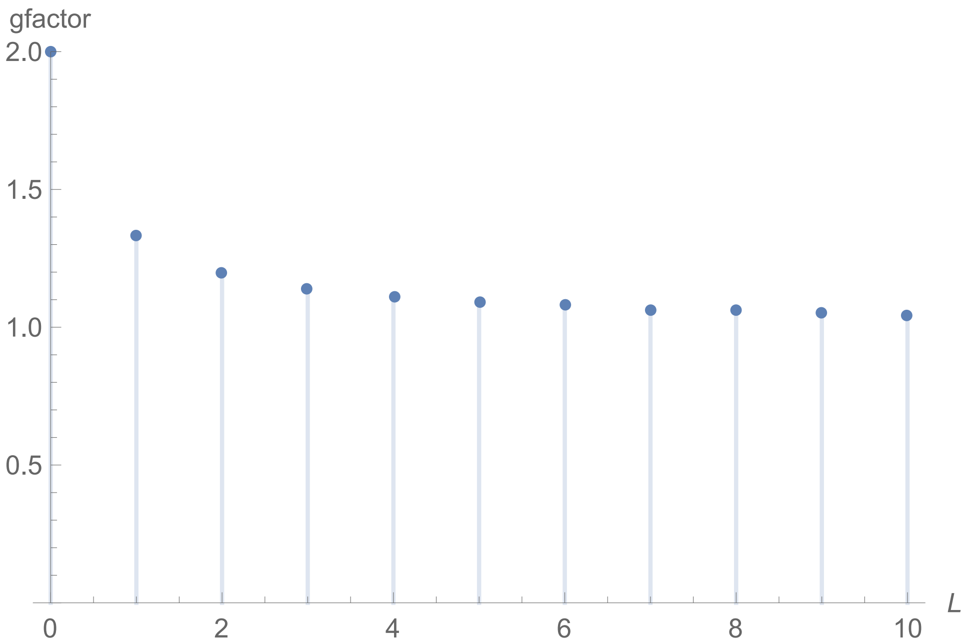
i. From triangle rule:
Largest: J = L + 1/2
Smallest: J = 1/2

ii. Units: Bohr magneton
Largest: μ = L + 2S = L + 1
Smallest: $\mu = g_J J$ = (2/3)(1/2) = 1/3 for the case of J = 1/2, as detailed below.

iii. In the case of L = 1, J = 1/2 (L and S anti-aligned), the g factor is minimized. For L > 1 the cancellation of L and S through anti-alignment is less than the contribution to the g factor from the increased L, therefore the g factor will increase. See the plots below calculated from the lande-g factor formula.

Largest: $g_J = g_S = 2$
Smallest: $g_J = 2/3$

L and S aligned g factor plot:



L and S anti-aligned g factor plot: