PSet 4 Solutions

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1 Ramsey Spectroscopy

1) This is the standard two-level atom transition probability

$$P_A = \frac{\Omega^2}{\Omega_g^2} \sin^2\left(\frac{\Omega_g \tau}{2}\right) \tag{1}$$

2) i) We start from the Schrodinger equation applied to $|\Psi(t)\rangle = a(t)|a\rangle + b(t)|b\rangle$ and get two coupled differential equations for the amplitudes $a(t)$ and $b(t)$

$$\begin{aligned} i\dot{a} &= -\frac{\delta}{2}a + \frac{\Omega}{2}b \\ i\dot{b} &= \frac{\Omega}{2}a + \frac{\delta}{2}b \end{aligned} \tag{2}$$

We solve them be substitution and get the following second order differential equation for $a(t)$:

$$\ddot{a} + \frac{\Omega^2}{4}a + \frac{\delta^2}{4}a = 0 \tag{3}$$

The solutions to the characteristic equation are:

$$\mathcal{E}_\pm = \pm\frac{\Omega_g}{2} \tag{4}$$

The time evolution of $a(t)$ and $b(t)$ is then:

$$\boxed{\begin{aligned} a(t) &= \alpha e^{i\Omega_g t/2} + \beta e^{-i\Omega_g t/2} \\ b(t) &= \alpha e^{i\Omega_g t/2}\left(\frac{\delta - \Omega_g}{\Omega}\right) + \beta e^{-\Omega_g t/2}\left(\frac{\delta + \Omega_g}{\Omega}\right) \end{aligned}} \tag{5}$$

We now have to solve for α and β, given the initial conditions $a(t_0) = a_0$ and $b(t_0) = b_0$.

$$\begin{aligned} a(t_0) = a_0 &= \alpha e^{i\Omega_g t_0/2} + \beta e^{-i\Omega_g t_0/2} \\ b(t_0) = b_0 &= \alpha e^{i\Omega_g t_0/2}\left(\frac{\delta - \Omega_g}{\Omega}\right) + \beta e^{-\Omega_g t_0/2}\left(\frac{\delta + \Omega_g}{\Omega}\right) \end{aligned} \tag{6}$$

From the first equation: $\beta e^{-i\Omega_g t_0/2} = a_0 - \alpha e^{i\Omega_g t_0/2}$. Substituting into the second equation gives:

$$b_0 = \alpha e^{i\Omega_g t_0/2}\left(-\frac{2\Omega_g}{\Omega}\right) + a_0\left(\frac{\delta+\Omega_g}{\Omega}\right) \tag{7}$$

So that we get α and β:

$$\boxed{\begin{aligned} \alpha e^{i\Omega_g t_0/2} &= a_0\frac{\delta+\Omega_g}{2\Omega_g} - b_0\frac{\Omega}{2\Omega_g} \\ \beta e^{-i\Omega_g t_0/2} &= a_0\frac{\Omega_g-\delta}{2\Omega_g} + b_0\frac{\Omega}{2\Omega_g} \end{aligned}} \tag{8}$$

Substituting eq-s (8) into eq-s (5), we finally get:

$$\begin{aligned} a(t+t_0) &= a_0\left(\frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) + \cos\left(\frac{\Omega_g t}{2}\right)\right) - b_0\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) \\ b(t+t_0) &= -a_0\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) + b_0\left(-\frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) + \cos\left(\frac{\Omega_g t}{2}\right)\right) \end{aligned} \tag{9}$$

ii) From eq (9), for $a_0 = 1$ and $b_0 = 0$, we let:

$$\begin{aligned} A = a(t+t_0) &= a_0\left(\frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) + \cos\left(\frac{\Omega_g t}{2}\right)\right) \\ B = b(t+t_0) &= -a_0\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) \end{aligned} \tag{10}$$

To express $|\Psi(t+t_0)\rangle$ as a density matrix, we use

$$\begin{aligned} \rho = |\Psi\rangle\langle\Psi| &= (A|a\rangle + B|b\rangle)\cdot(\langle a|A^* + \langle b|B^*) \\ &= |A|^2|a\rangle\langle a| + AB^*|a\rangle\langle b| + A^*B|b\rangle\langle a| + |B|^2|b\rangle\langle b| \\ &= \rho_{aa}|a\rangle\langle a| + \rho_{ab}|a\rangle\langle b| + \rho_{ba}|b\rangle\langle a| + \rho_{bb}|b\rangle\langle b| \end{aligned} \tag{11}$$

We get the following expressions for the density matrix entries:

$$\begin{aligned} \rho_{aa}(t+t_0) &= \cos^2\left(\frac{\Omega_g t}{2}\right) + \frac{\delta^2}{\Omega_g^2}\sin^2\left(\frac{\Omega_g t}{2}\right) \\ \rho_{bb}(t+t_0) &= \frac{\Omega^2}{\Omega_g^2}\sin^2\left(\frac{\Omega_g t}{2}\right) \\ \rho_{ab}(t+t_0) &= -\frac{\delta\Omega}{\Omega_g^2}\sin^2\left(\frac{\Omega_g t}{2}\right) + \frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right)\cos\left(\frac{\Omega_g t}{2}\right) \\ \rho_{ba}(t+t_0) &= \rho_{ab}(t+t_0)^* \end{aligned} \tag{12}$$

iii) We now want to use the Hamiltonian time evolution of the density matrix to get the same result. The Hamiltonian can be expressed in terms of the 2×2 Pauli matrices:

$$H = -\frac{\hbar\delta}{2}\sigma_z + \frac{\hbar\Omega}{2}\sigma_x \tag{13}$$

We want evolution of the form $e^{-iHt/\hbar} = e^{i\alpha(\hat{n}\cdot\vec{\sigma})}$. In our case:

$$\sigma = (\sigma_x, \sigma_y, \sigma_z) \tag{14}$$

$$\alpha\hat{n} = (\Omega, 0, -\delta)\frac{t}{2} = \frac{\Omega_g t}{2}\frac{(\Omega, 0, \delta)}{\Omega_g} \tag{15}$$

$$\rightarrow \alpha = -\frac{\Omega_g t}{2}, \quad \hat{n}\cdot\sigma = \frac{-\delta\sigma_z + \Omega\sigma_x}{\Omega_g} \tag{16}$$

The evolution operator is then:

$$U(t) = \mathbf{I}\cos\left(\frac{\Omega_g t}{2}\right) - i\left(\frac{-\delta\sigma_z + \Omega\sigma_x}{\Omega_g}\right)\sin\left(\frac{\Omega_g t}{2}\right) \tag{17}$$

The atom is initially in $\Phi(t_0) = |a\rangle$, so that

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{18}$$

The evolution under this Hamiltonian is:

$$\rho(t + t_0) = U(t)\rho_0 U(t)^\dagger \tag{19}$$

$$= \begin{bmatrix} \cos\left(\frac{\Omega_g t}{2}\right) + \frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) & -\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) \\ -\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) & \cos\left(\frac{\Omega_g t}{2}\right) - \frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\left(\frac{\Omega_g t}{2}\right) - \frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) & -\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) \\ -\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) & \cos\left(\frac{\Omega_g t}{2}\right) + \frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) \end{bmatrix} \tag{20}$$

Multiplying it out gives eq-s (12).

iv) To get the full evolution after the Ramsey sequence, we need the free evolution of the density matrix for $\Omega = 0$. The transformation matrix for the free evolution is

$$U_f(T) = \mathbf{I}\cos\left(\frac{\delta T}{2}\right) + i\sin\left(\frac{\delta T}{2}\right)\sigma_z \tag{21}$$

The full evolution is:

$$\rho(\tau) \equiv \rho_1 = U(\tau)\rho_0 U^\dagger(\tau) \tag{22}$$

$$\begin{aligned} \rho(\tau + T) &\equiv \rho_2 = U_f(T)\rho_1 U_f^\dagger(T) \\ \rho(2\tau + T) &\equiv \rho_3 = U(\tau)\rho_2 U^\dagger(\tau) \end{aligned} \tag{23}$$

We see that ρ_1 is just eq-s (12) above with $t_0 = 0$, $t = \tau$. After the free evolution:

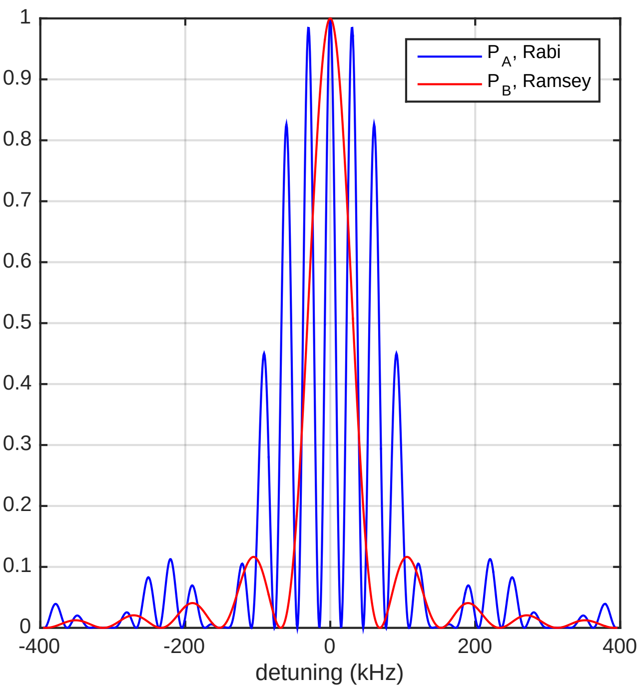
$$\rho_2 = \begin{bmatrix} \rho_{aa}(\tau) & \rho_{ab}(\tau)e^{i\delta T} \\ \rho_{ba}(\tau)e^{-i\delta T} & \rho_{bb}(\tau) \end{bmatrix} \tag{24}$$

$$= \begin{bmatrix} \cos^2\left(\frac{\Omega_g t}{2}\right) + \frac{\delta^2}{\Omega_g^2}\sin^2\left(\frac{\Omega_g t}{2}\right) & -\frac{\delta\Omega}{\Omega_g^2}\sin^2\left(\frac{\Omega_g t}{2}\right) + \frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right)\cos\left(\frac{\Omega_g t}{2}\right)e^{i\delta T} \\ -\frac{\delta\Omega}{\Omega_g^2}\sin^2\left(\frac{\Omega_g t}{2}\right) - \frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right)\cos\left(\frac{\Omega_g t}{2}\right)e^{-i\delta T} & \frac{\Omega^2}{\Omega_g^2}\sin^2\left(\frac{\Omega_g t}{2}\right) \end{bmatrix}$$

Finally, we only need the ρ_{bb} element of ρ_3, which gives the transition probability $P_B = \rho_{bb,3}$

$$P_B = \frac{4\Omega^2}{\Omega_g^2}\sin^2\left(\frac{\Omega_g \tau}{2}\right)\left[\cos\left(\frac{\Omega_g \tau}{2}\right)\cos\left(\frac{\delta T}{2}\right) - \frac{\delta}{\Omega_g}\sin\left(\frac{\Omega_g \tau}{2}\right)\sin\left(\frac{\delta T}{2}\right)\right]^2 \tag{25}$$

3) P_A is maximized for $\Omega_g \tau = \pi$ and P_B is maximized for $\Omega_g \tau = \pi/2$.
To decrease the linewidth in the Rabi case, the interaction time τ has to be increased (still, for max signal, we need $\Omega_g \tau = n\pi$). In the Ramsey case, both τ and T can be made longer to decrease the linewidth. If there are inhomogeneities in the EM field, we can use the Ramsey method with $\tau \ll T$, so that the effect of the inhomogeneities is minimized but the linewidth is still small.



4) Starting from P_B, we can neglect the second term in the brackets because it has the prefactor $\frac{\delta}{\Omega} \ll 1$.

$$P_B = \frac{4\Omega^2}{\Omega_g^2}\sin^2\left(\frac{\Omega_g \tau}{2}\right)\cos^2\left(\frac{\Omega_g \tau}{2}\right)\cos^2\left(\frac{\delta T}{2}\right) \tag{26}$$

In this limit $\Omega_g \approx \Omega$, so

$$\begin{aligned} P_B &= 4\sin^2\left(\tfrac{\Omega\tau}{2}\right)\cos^2\left(\tfrac{\Omega\tau}{2}\right)\cos^2\left(\tfrac{\delta T}{2}\right) \\ &= \sin^2\left(\Omega\tau\right)\cos^2\left(\tfrac{\delta T}{2}\right) \\ &= \sin^2\left(\tfrac{\pi}{2}\right)\cos^2\left(\tfrac{\delta T}{2}\right) \\ &= \cos^2\left(\tfrac{\delta T}{2}\right) \end{aligned} \tag{27}$$

2 Ramsey Spectroscopy with Decoherence

1) For $\delta \ll \Omega$ and for $\Omega_g\tau = \pi/2$, the interaction Hamiltonian simplifies to

$$U_{\pi/2} = \mathbf{I}\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\sigma_x = \frac{1}{\sqrt{2}}\left(\mathbf{I} - i\sigma_x\right)$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \tag{28}$$

After the first $\pi/2$ pulse, we get

$$\rho_{1,\Gamma} = U_{\pi/2}\rho_0 U^\dagger_{\pi/2} = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \tag{29}$$

For the free evolution without decay, we use eq (21)

$$\rho_2 = U_f(T)\rho_{1,\Gamma}U_f^\dagger(T) = \frac{1}{2}\begin{bmatrix} 1 & ie^{i\delta T} \\ -ie^{-i\delta T} & 1 \end{bmatrix} \tag{30}$$

We modify the off-diagonal entries to account for the decay:

$$\rho_{2,\Gamma} = \frac{1}{2}\begin{bmatrix} 1 & ie^{i\delta T}e^{-\Gamma T} \\ -ie^{-i\delta T}e^{-\Gamma T} & 1 \end{bmatrix} \tag{31}$$

Finally, we after the second $\pi/2$ pusle, we get:

$$\rho_{3,\Gamma} = U_{\pi/2}\rho_{2,\Gamma}U^\dagger_{\pi/2} = \begin{bmatrix} \frac{1}{2}\left(1 - \cos\left(\delta T\right)e^{-\Gamma T}\right) & -\frac{1}{2}\sin\left(\delta T\right)e^{-\Gamma T} \\ -\frac{1}{2}\sin\left(\delta T\right)e^{-\Gamma T} & \frac{1}{2}\left(1 + \cos\left(\delta T\right)e^{-\Gamma T}\right) \end{bmatrix} \tag{32}$$

The probability of transition is the ρ_{bb} entry:

$$P_1 = \frac{1}{2}\left(1 + \cos\left(\delta T\right)e^{-\Gamma T}\right) \tag{33}$$

2) We need to integrate over the detunings:

$$P_2(t) = \int d\delta P_1(\delta)\frac{1}{\sqrt{2\pi\Gamma_2^2}}e^{-\frac{(\delta-\delta_0)^2}{2\Gamma_2^2}} \tag{34}$$

$$P_2 = \frac{1}{2}\left(1 + \cos\left(\delta T\right)e^{-\Gamma T}e^{-\Gamma_2^2 T^2/2}\right) \tag{35}$$

3) We take our result eq (31) with $T \to T_A$ for the density matrix after the first $\pi/2$ pulse and the first free evolution of duration T_A.

$$\tilde{\rho}_{2,\Gamma} = \frac{1}{2}\begin{bmatrix} 1 & ie^{i\delta T_A}e^{-\Gamma T_A} \\ -ie^{-i\delta T_A}e^{-\Gamma T_A} & 1 \end{bmatrix} \tag{36}$$

During the π pulse the evolution is

$$U_\pi(t) = \mathbf{I}\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\sigma_x = -i\sigma_x \tag{37}$$

so that after the π pulse we get:

$$\tilde{\rho}_{3,\Gamma} = \frac{1}{2}\begin{bmatrix} 1 & -ie^{-i\delta T_A}e^{-\Gamma T_A} \\ ie^{i\delta T_A}e^{-\Gamma T_A} & 1 \end{bmatrix} \tag{38}$$

All we have done is to swap the diagonal entries, i.e. time-reverse the evolution so far.
During the next free evolution of duration T_B, with no damping, we get:

$$\tilde{\rho}_4 = U_f(T_B)\tilde{\rho}_{3,\Gamma}U_f^\dagger(T_B) = \frac{1}{2}\begin{bmatrix} 1 & -ie^{-i\delta(T_A-T_B)}e^{-\Gamma T_A} \\ ie^{i\delta(T_A-T_B)}e^{-\Gamma T_A} & 1 \end{bmatrix} \tag{39}$$

Again, we tag the damping $e^{-\Gamma T_B}$ to the off-diagonal elements:

$$\tilde{\rho}_{4,\Gamma} = \frac{1}{2}\begin{bmatrix} 1 & -ie^{-i\delta(T_A-T_B)}e^{-\Gamma(T_A+T_B)} \\ ie^{i\delta(T_A-T_B)}e^{-\Gamma(T_A+T_B)} & 1 \end{bmatrix} \tag{40}$$

After the last $\pi/2$ pulse we get:

$$\tilde{\rho}_{5,\Gamma} = U_{\pi/2}\tilde{\rho}_{4,\Gamma}U_{\pi/2}^\dagger = \begin{bmatrix} \frac{1}{2}\left(1 + e^{-\Gamma(T_A+T_B)}\cos\left(\delta(T_A - T_B)\right)\right) & -\frac{1}{2}e^{-\Gamma(T_A+T_B)}\sin\left(\delta(T_A - T_B)\right) \\ -\frac{1}{2}e^{-\Gamma(T_A+T_B)}\sin\left(\delta(T_A - T_B)\right) & \frac{1}{2}\left(1 - e^{-\Gamma(T_A+T_B)}\cos\left(\delta(T_A - T_B)\right)\right) \end{bmatrix} \tag{41}$$

The transition probability $\tilde{P}_3$ is the ρ_{bb} entry:

$$\tilde{P}_3 = \frac{1}{2}\left(1 - e^{-\Gamma(T_A+T_B)}\cos\left(\delta(T_A - T_B)\right)\right) \tag{42}$$

Finally, we have to account for inhomogeneous damping. We integrate over detunings, similarly to part 2):

$$P_3 = \int d\delta \tilde{P}_3(\delta)\frac{1}{\sqrt{2\pi\Gamma_2^2}}e^{-\frac{(\delta-\delta_0)^2}{2\Gamma_2^2}} \tag{43}$$

This gives:

$$P_3 = \frac{1}{2}\left(1 - \cos(\delta(T_A - T_B)e^{-\Gamma(T_A+T_B)}e^{-(T_A-T_B)^2\Gamma_2^2/2})\right) \tag{44}$$

For $T_A = T_B$, the "echo condition", the inhomogeneous broadening is completely eliminated. However, the sensitivity to the detuning δ is also eliminated. Therefore, the echo technique helps you to distinguish and measure the homogeneous broadening, but not to perform high resolution spectroscopy.

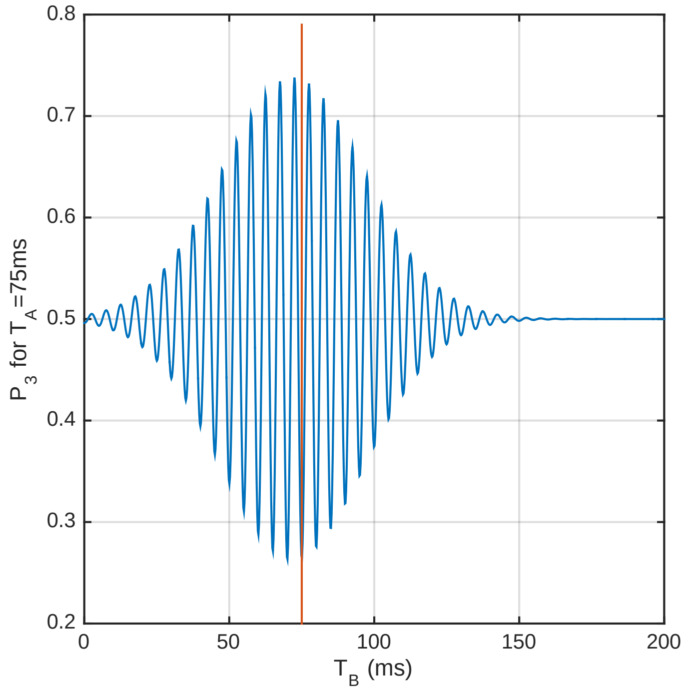


Figure 1: A plot of an echo pulse $P_3(T_B)$ plotted vs T_B with $T_A = 75\ ms$. The vertical line marks $T_B = T_A$.

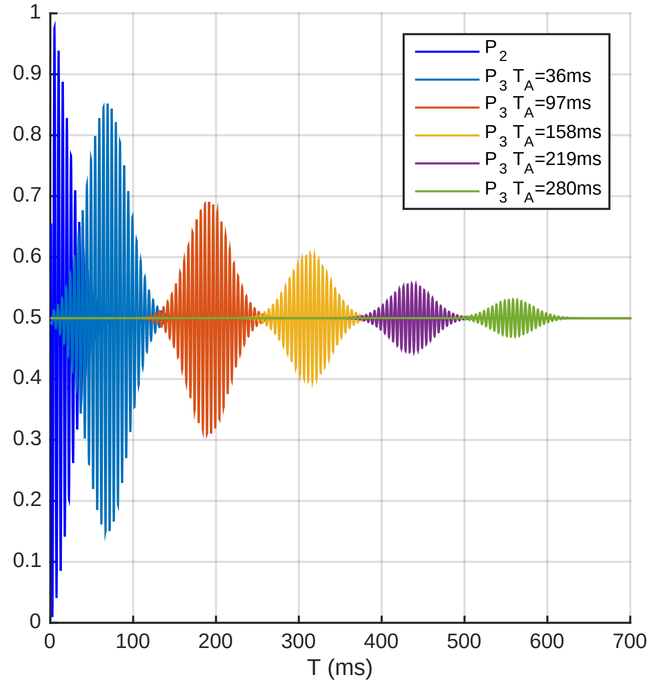


Figure 2: A plot of the non-echo $P_2(T)$ and $P_3(T)$ now plotted vs T. The non-echo pulse decays as the free evolution time is increased due to both homogeneous and inhomogeneous mechanisms. The echo pulses are still visible for longer times T because the echo sequence can cancel inhomogeneous decay if T_A is chosen in the vicinity of T_B. Since we have picked $\Gamma < \Gamma_2$, the inhomogeneous decay dominates and the effect of having the time-reversing $\pi-$pulse can be observed.

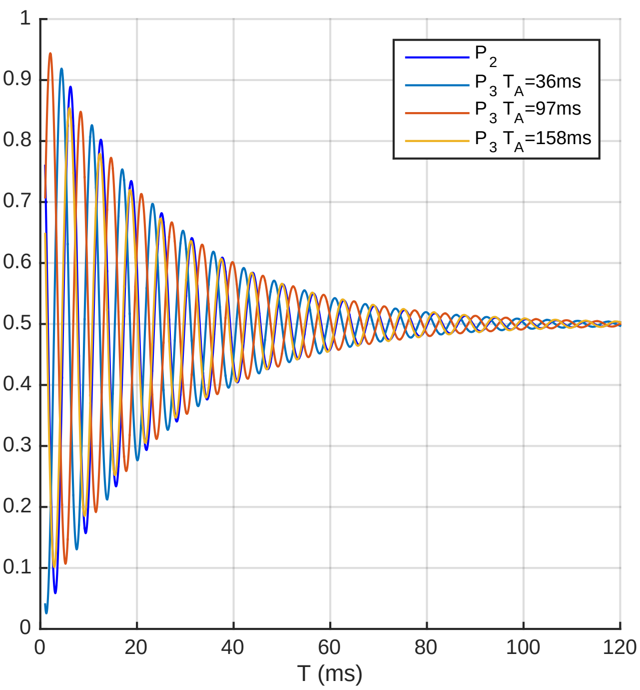


Figure 3: If we swap the values of Γ and Γ_2, i.e. $\Gamma = 40\ s^{-1}$ and $\Gamma_2 = 5\ s^{-1}$, we see that both $P_2(T)$ and $P_3(T)$ decay with the same time constant because now the homogeneous decay dominates.