PROBLEM SET ON SUPERRADIANT LIGHT SCATTERING IN A BOSE-EINSTEIN CONDENSATE

This problem set addresses a seemingly simple problem. What happens when non-interacting atoms in a single quantum state (a Bose-Einstein condensate) scatter light from a laser beam. This is the physics of Rayleigh scattering, the physics of the blue sky. However, the Bose-Einstein condensate is very cold, and a scattered photon has deposited momentum in the condensate in form of a recoiling atom. This recoiling atom can interfere with the condensate at rest and create a periodic density modulation — this is like a diffraction grating, which will now start to diffract the laser beam, thus enhancing and amplifying the scattering process. This self-amplifying emission of photons into an initially empty mode is an ideal realization of Dicke superradiance in an extended sample.

In this problem set, you will derive the gain equation both semiclassically (using the concept of the density modulation as diffraction grating) and quantum mechanically, using the concepts of a two-photon process. The gain has to be stronger than a loss rate — you will identify the relevant loss rate to be equal to the Doppler width of the two-photon process. In Dicke superradiance, the crucial element is that the radiating system "remembers" how many photon it has emitted and will now emit at an accelerated rate. In this homework problem you will identify how the condensate remembers the scattered photons — via the recoiling atoms, which have imprinted a density modulation into the BEC.

In the second problem, you will realize that this system, below the threshold for superradiance, can act as an amplifier for light and as an amplifier for matter waves. Superradiance starts spontaneously, whereas for amplification, you start by seeding the process with a weak laser beam in the scattering mode, or with atoms which have already the recoil momentum. In either case, you will find out how Rayleigh scattering is now the microscopic process behind the amplification. Depending on the damping rate for light and for atoms, you will retrieve the limit of the conventional laser (bosonic stimulation by photons) and the so-called bad cavity limit where all amplification is via superradiance. Finally, you will analyze that the amplified light propagates with an extremely slow group velocity (observed to be 1 m/s).

The whole problem has very simple ingredients: Atoms at rest scatter light from a laser beam into a selected mode. However, by keeping track of the recoiling atom, we find enormous richness: superradiance, amplification of light and atoms, slow light. This is textbook style physics, but it was all discovered and experimentally explored at MIT between 1999 and

2001. Most of these phenomena were not predicted — a nice example how we learn by observing Nature!

References:

S. Inouye, A.P. Chikkatur, D.M. Stamper-Kurn, J. Stenger, D.E. Pritchard, and W. Ketterle: Superradiant Rayleigh scattering from a Bose-Einstein condensate. Science 285, 571-574 (1999).

S. Inouye, T. Pfau, S. Gupta, A.P. Chikkatur, A. Görlitz, D.E. Pritchard, and W. Ketterle: Phase-coherent amplification of atomic matter waves. Nature 402, 641-644 (1999).

S. Inouye, R.F.Löw, S. Gupta, T. Pfau, A. Görlitz, T. L. Gustavson, D.E. Pritchard and W. Ketterle: Amplification of Light and Atoms in a Bose-Einstein Condensate. Phys. Rev. Lett. 85, 4225-4228 (2000), arXiv:cond-mat/0006455.

W. Ketterle and S. Inouye: Does matter wave amplification work for fermions? Phys. Rev. Lett. 86, 4203-4206 (2001), arXiv:cond-mat/0008232.

W. Ketterle and S. Inouye: Collective enhancement and suppression in Bose-Einstein condensates. Compte rendus de l'académie des sciences, Série IV - Physique Astrophysique, vol. 2, pp. 339-380 (2001); arXiv:cond-mat/0101424.

Further references: D. Schneble, Y. Torii, M. Boyd, E.W. Streed, D.E. Pritchard, and W. Ketterle: The Onset of Matter-Wave Amplification in a Superradiant Bose-Einstein Condensate. Science 300, 475-478 (2003); published online in Science Express on March 27, 2003.

D. Schneble, G.K. Campbell, E.W. Streed, M. Boyd, D.E. Pritchard, and W. Ketterle: Raman Amplification of Matter Waves. Phys. Rev. A 69, 041601(R) (2004).

1. Problem 1

An ideal Bose-Einstein Condensate (BEC) of cylindrical shape with length l and diameter d is illuminated with a pump laser beam along $\hat{\mathbf{e}}_x$, orthogonal to the long axis of the BEC. The laser polarization is along the y-axis. Atoms can scatter photons in all directions (Rayleigh scattering), changing their momentum from zero (BEC!) to the (two-photon) recoil momentum $\vec{\mathbf{q}}$. If the laser's wave number is k, and the direction of the emitted photon is $\hat{\mathbf{e}}_{\text{out}}$, then the momentum transfer is

$$\vec{\mathbf{q}} = k\,(\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_{\text{out}}) \tag{1}$$

Note that the change of momentum of the scattered photon due to the loss of the recoil energy is negligible.

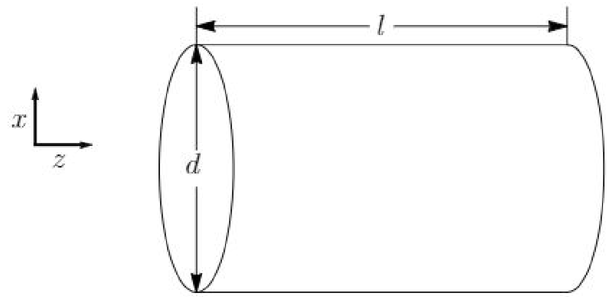

FIGURE 1. Geometry of the condensate

Let's first consider the physics behind scattering with a particular momentum transfer $\vec{\mathbf{q}}$, i.e. we select one mode. If you want you can assume that there is a cavity which allows light scattering only into this direction, and we can focus on only two modes for the atoms (the condensate and the atoms with recoil momentum $\hbar\vec{\mathbf{q}}$ and two modes for the light (the laser beam and the photons scattered into a certain direction).

1.1. **Classical derivation of the gain equation.** In several steps, you will derive the gain equation for the population N_q of recoiling atoms based on the picture that the recoiling atoms interfere with the condensate at rest and create a density modulation. This density modulation diffracts light, thus enhancing the scattering of light with momentum transfer $\hbar q$.

1.1.1. *Constructive interference factor.* For single atom, the Rayleigh scattering rate is R_{Rayleigh}. For N atoms spread out homogeneously, the amplitudes add up in a random manner. So there is no interference, and the total scattering rate is NR_{Rayleigh}, i.e. N times the single-atom scattering rate. Now, when N atoms are arranged in a co-sinusoidal density distribution with density proportional $N\cos(qr)$, what is the enhancement factor for scattering light with a momentum transfer $\hbar q$?

1.1.2. *Density modulation in BEC.* Now consider the density modulation in a BEC created by N_q recoiling atoms with momentum $\hbar q$ against the background of N_0 atoms with zero momentum. What is the enhancement factor of the Rayleigh scattering rate due to this density modulation?

Hint: A BEC with average number N is described by the macroscopic wavefunction $\sqrt{N}e^{i\phi(r)}$, where $\phi(r)$ is the phase of the BEC. The gradient of the phase is related to the momentum of the atoms. For simplicity, the volume of the condensate is assumed to be 1, i.e. number and density are the same.

Note: In the next step, we consider that the atoms with momentum $\hbar q$ are created by previous scattering events.

1.1.3. *Angular factor.* The dipole emission intensity pattern has $\sin^2(\theta)$ angular dependence, where θ is the angle between the dipole vector and the vector between the dipole and the measurement point. Find the angular dependence of the emission into some direction j with the appropriate normalization factor. Use θ_j as the angle variable.

1.1.4. *Mode volume.* Based on our work so far, we can calculate how much light is scattered per solid angle (in other words, into an infinitesimal angular interval). But when the scattering source has finite dimensions, the scattering amplitudes that sum up coherently can be grouped into a countable number of modes, each of which has a finite solid angle Ω_j, where j is the index for such modes. Find the dependence of Ω_j on the wavelength of light λ and the dimensions of the BEC. Which direction of the scattered light has the largest Ω_j?

Hints: There are two ways to look at this problem: the first one is to find the diffraction limit, and the other is to use the Heisenberg uncertainty due to finite dimensions. They should give the same answers. You may also think about a disk of area A coherently emitting light.

.

1.1.5. *Gain equation for N_q.* We have seen that the existence of atoms with momentum $\hbar q$ leads to an enhancement of Rayleigh scattering and therefore to an accelerated increase of N_q. Using the previous answers, find the gain coefficient G_q in the gain equation $\dot{N}_q = G_q N_q$. Use R for the single atom Rayleigh scattering rate, N_0 for the average number of zero momentum atoms, θ for the angle between the polarization of the pump light and the emission direction, and Ω for the emission mode solid angle. Assume emission along the condensate axis.

Note: Since the solid angle Ω_j is largest for emission along the cylindrical axis, scattering into this mode, called endfire mode, has the largest gain. Since the gain enters an exponent, the amplified Rayleigh scattering will predominantly go into the two endfire modes, even without mode selection or a surrounding cavity. In the remainder of the problem set, we will only consider gain for the endfire modes.

1.2. **Quantum-mechanical derivation of the gain equation.** Now we derive the gain equation quantum mechanically, using the "four-wave mixing" Hamiltonian, coupling two optical and two atomic fields:

$$\mathcal{H}' = \sum_{k,l,n,m} C_{klnm} \hat{c}_l^\dagger \hat{a}_n^\dagger \hat{c}_k \hat{a}_m \delta_{l+n-k-m}$$

This Hamiltonian is very intuitive: It removes an atom in the condensate and a photon from the pump laser, and creates a recoiling atom and a scattered photon. This four-wave mixing Hamiltonian can we derived from the atom-light Hamiltonian in the rotating wave approximation, through adiabatic elimination of the excited state (i.e. setting the time derivative of excited state amplitudes to zero, so that there is only single equation of motion for the ground state). See separate document if you are interested.

1.2.1. *Fermi's golden rule.* Use Fermi's golden rule to calculate the scattering rate per unit energy interval and solid angle. Ignore depletion of the BEC and the pump light, so that the corresponding operators may be replaced by c-numbers. Assume the number of photons in the scattered light is negligible (i.e. the emitted photon escapes the system very fast).

1.2.2. *Integration over density of states.* What is the density of states for photons of a given polarization? Express your answer in volume V, frequency ω, reduced Planck constant $\hbar$ and speed of light c.

1.2.3. *Expression for the Rayleigh scattering rate.* We need to express the constant $|C|$ in terms of microscopic parameters. This can be done by using the results of the derivation of the four-wave mixing Hamiltonian from the standard electric dipole coupling between atoms and light. Here we suggest an equivalent approach using a result from two-photon processes.

First, remember from discussion of field quantization (e.g. Jaynes-Cummings Hamiltonian introduced in Module 3) that the number of photons in volume V can be expressed in terms of field strength E as

$$n_k = \frac{\varepsilon_0 E^2}{2\hbar\omega} V$$

The relationship between $|C|$ and two-photon Rabi frequency Ω_R is

$$\left(\frac{\hbar\Omega_R}{2}\right)^2 = \left(\frac{d^2 E_1 E_2}{\hbar\Delta}\cos\phi\right)^2 = |C|^2 n_1 n_2$$

(note $|C|$ should only contain expressions inherent to atomic structure) where the first equality is obtained from second-order time-dependent perturbation theory. Putting all this together, obtain expression for $|C|^2$ using the dipole matrix element d, atomic frequency ω_0, detuning from resonance Δ, and angle between polarizations of the two beams ϕ. Use ϵ_0 for permittivity and V for volume.

1.2.4. *Gain equation.* Finally, obtain the quantum-mechanical gain equation for N_q, using the same definition of mode as in the classical derivation. You should use the fact that in rotating wave approximation, R is given by the product of the natural linewidth and admixture of the excited state (as in first order perturbation theory). Note that $\cos\phi = \sin\theta$, where ϕ is the angle between polarizations of the two-photon beams and θ is the angle as defined in the classical derivation.

1.3. **Gain equation with loss.** Now add a loss term for N_q:

$$\dot{N}_q = G_q(N_q + 1) - \Gamma_{2,q} N_q$$

$$G_q = RN_0 \frac{\sin^2\theta}{8\pi/3} \Omega_q$$

(a) What is the characteristic rate for loss of N_q? Use v for the velocity of the recoiling atoms.

(b) Show that the loss rate Γ_q is the Doppler width of the two-photon Bragg transition, driven by the pump laser and a beam along the long axis of the cloud.

(c) Find the threshold for superradiance by solving the growth equation.

Hint: For the Doppler width, first consider the momentum and energy conservations for a free particle with momentum $\hbar k$ receiving momentum transfer $\hbar q$. Denote the initial momentum of an atom by $\hbar k$, and the final momentum by $\hbar k_f$

.

1.4. **Dicke superradiance.** In Dicke superradiance, what is the rate for emission of the Nth photon (N smaller than the total atom number)

1.5. **Superradiance in two pictures.** Is superradiance enhanced spontaneous emission, or can it be regarded as an amplification of spontaneously emitted photons (by stimulated emission)?

1.6. **Superradiance by thermal atoms.** What is different if we replace the BEC by a thermal cloud (i.e. Boltzmann gas)? Do we still get superradiance?

2. Problem 2: Superradiance and Lasing in a Bose-Einstein Condensate

With a cylindrically shaped BEC, superradiance can result in the spontaneous creation of a laser beam in the "endfire" mode, and a matter wave "beam" with recoil momentum $\hbar q$. If we "seed" the superradiant system with either a laser beam along the long axis, or a matter wave beam with momentum q, we realize an amplifier for light, or for matter waves, respectively. In other words, we want to use the same system we have used to describe superradiance now as an amplifier, either for light, or for matter waves. In the following, we will find the gain for such amplifiers and see that the gain diverges at the threshold for superradiance.

Ref: S. Inouye, R.F.Löw, S. Gupta, T. Pfau, A. Görlitz, T. L. Gustavson, D. E. Pritchard and W. Ketterle: Amplification of Light and Atoms in a Bose-Einstein Condensate. Phys. Rev. Lett. 85, 4225-4228 (2000).

2.1. **Heisenberg equation of motion.** Use the four wave mixing Hamiltonian and derive the equation of motion for the endfire mode of the photons, and the atomic recoil mode. Ignore depletion of the condensate and the pump beam. Use a for the atomic field and c for the light field to be amplified. Hint: Treat the operators for atoms in the condensate and photons in the pump beam as c-numbers which is valid in the limit of large occupation numbers.

2.2. **Differential equation for atom/light field.** Replace the operators by c-numbers (usually valid for macroscopically occupied modes), add phenomenological damping rates and source terms (input fields), and find coupled differential equations for the fields a and c^*. Use a_0 and c_0 as the equilibrium field values where the pump rate and decay rate balances out.

2.3. **Optical and matter wave gain.** Find the optical and matter wave gain (below threshold for exponential growth). The matter wave gain is defined as $g = a(t \to \infty)/a_0$ (with $c_0 = 0$, i.e. no light field input). The optical gain is defined as $g_c = c(t \to \infty)/c_0$ (with $a_0 = 0$, i.e. no matter field input):

2.4. **Two limiting cases.** Find simplified gain equations in the two limits, where the optical damping is much larger than the matter wave damping, and vice versa.

2.5. Express the gain calculated in part 3 in terms of the gain and loss derived in the first part of the problem on superradiance.

2.6. **Slow group velocity.** Finally, we want to show that the amplified optical wave has a very slow group velocity ("slow light"). This is a way to obtain slow light different from EIT discussed in class.

In the previous part, we found the gain for optical amplification. The linewidth of this gain is the Doppler width of the two-photon transition. Therefore, assume that you have a Lorentzian gain profile with gain g' and linewidth Γ. What is index of refraction of such a medium, and what is the group velocity?

Hints: Review discussion of atomic polarizability and index of refraction. (Texts should be available in Module 2, week 3) `https://amowiki.odl.mit.edu/index.php/Atoms_in_electric_fields`

2.7. Find a relation between the optical gain and the pulse delay.