3/19/2014

Atom-light interaction

Outline:

TODAY → : Coupling matrix element

. narrowband & broadband

. quantized em Field

Results of classical E&M & Lagrangian formalism:

$$p_{can} = p_{kin} + \frac{q}{c}A$$

(vector potential)

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + V(r)$$

E_{kin}

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$

$$[\vec{A}, \vec{p}] = 0$$

$\vec{A}, \vec{r}$ commute

$$H = \underbrace{\frac{p^2}{2m} + V(r)}_{H_0} + \underbrace{\frac{e}{mc}\vec{p}\cdot\vec{A}}_{H_{INT}} + \underbrace{\frac{e^2}{2mc^2}\vec{A}^2}_{H^{(2)}}$$

Atom

6

a

$\begin{pmatrix} \cdot & x \\ x & \cdot \end{pmatrix}$

$\langle 1|\hat{O}|2\rangle$

E_d

$\langle 1|\vec{r}|2\rangle$

H_{12}

Γ_{12}

$$H_{INT} = \frac{e A_0}{mc} \hat{e} \cdot \vec{p}\, e^{i\vec{k}\vec{r} - i\omega t}$$

↑ plane wave

$$\langle b | H_{INT} | a \rangle = H_{ba}\, e^{-i\omega t}$$

$$H_{ba} = \frac{e A_0}{mc} \hat{e} \langle b | \vec{p}\, e^{i\vec{k}\vec{r}} | a \rangle$$

dipole approx.: $\vec{A}(\vec{r}) \approx \vec{A}(0)$

$e^{i\vec{k}\vec{r}} \to 1 + i\vec{k}\vec{r} - \frac{1}{2}(\vec{k}\vec{r})^2$

$\vec{k}\cdot\vec{r} \ll 1$ (long-wavelength approximation)

leading term: dipole

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$H_{ba} = -\frac{ieE_0}{m\omega} \hat{e} \langle b | \vec{p} | a \rangle = \underbrace{\frac{i\omega}{c} A_0}_{E_0} \hat{e}$$

$$\vec{p} = \frac{m}{i\hbar} [\vec{r}, H_0]$$

$$\Rightarrow \langle b|\vec{p}|a\rangle = im\omega_{ba}\langle b|\vec{r}|a\rangle$$

$$H_{ba} = \underbrace{eE_0\hat{e}\langle b|\vec{r}|a\rangle}_{-D_{ba}}\underbrace{\frac{\omega_{ba}}{\omega}}_{\simeq 1}$$

near resonance

$$H_{INT}^{Dipole} = e\vec{E}\vec{r}$$

Higher-order radiation processes

$$H_{ba} = \frac{eA}{mc}\langle b|p_z\left(1 - i\underline{kx} + \tfrac{1}{2}(ikx)^2 - \right)|a\rangle$$

$$A(r) = A\hat{z}\,e^{ikx}$$

$$kr = \frac{\hbar\omega}{\hbar c}a_0 = \frac{e^2}{\hbar c} = \alpha$$

$$\hbar\omega = \frac{e^2}{a_0}$$

2nd term: $p_z x = \underbrace{\frac{1}{2}(p_z x - zp_x)}_{-\frac{1}{2}\hbar L_y} + \frac{1}{2}(p_z x + zp_x)$

μ_0

$$\frac{-ie\hbar Ak}{2mc}\langle b|L_y|a\rangle = \vec{\beta}\underline{\mu_B}\langle b|\underline{\vec{L}}|a\rangle$$

$\vec{\mu}$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad B = -i k A \hat{y}$$

$$H_{INT}(M1) = \mu_B \vec{B} \langle b | \vec{L} + g_s \vec{S} | a \rangle$$

"magnetic dipole transition"

$$= \vec{\mu} \cdot \vec{B}$$

Second term:

$$\frac{1}{2}(p_z x + z p_x) = \frac{m}{2i\hbar}\left([z, H_0]x + z[x, H_0]\right)$$

$$[r, H_0] = i\hbar p/m$$

$$= \frac{m}{2i\hbar}\left(-H_0 z x + z x H_0\right)$$

Contributes to H_{1a}

$$\frac{ieA}{mc}\frac{2m}{2\hbar i}\langle b | -H_0 z x + z x H_0 | a \rangle$$

$$= -\frac{eA k}{2c}\frac{E_b - E_a}{\hbar}\langle b | z x | a \rangle$$

$$= \frac{ie E \omega}{2c}\langle b | z x | a \rangle$$

$E = i k A$

$$H_{INT}(E2) = \frac{ie\omega}{2c} \langle b | zx | a \rangle \epsilon$$

$$\langle b | \underbrace{\vec{r}\,\vec{r}}_{\text{tensor}} | a \rangle$$

$$H_{INT}^{(2)} = \underbrace{H_{INT}(M1)}_{\text{real}} + \underbrace{H_{INT}(E2)}_{\text{imaginary}}$$

$\Rightarrow$ no interference

$$\left| H_{INT}^{(2)} \right|^2 = \left| H_{INT}(M1) \right|^2 + \left| H_{INT}(E2) \right|^2$$

Summary:

		Operator	Parity
Electric dipole	E1	$-e\vec{r}$	$-$
Magnetic Dipole	M1	$-\mu_B(\vec{L} + g_s\vec{S})$	$+$
Electric Quadrupole	E2	$-e\vec{r}\,\vec{r}$	$+$

M1, E2 "forbidden" transitions

$\alpha^2 = 5 \times 10^{-5}$ weaker than an allowed E1 transition

Selection rules

Forbidden transitions are weaker by α^n require higher approximation

Multipole exp. E2, M1
Multi-photon processes
relativistic effects (e.g. S→T in Hg)
hyperfine interactions

Matrix elements

$$\langle n J M | T_{l,m} | n' J' M' \rangle =$$

even operator = Sum of Spherical tensors

spherical tensor of rank l (transforms under rotations like the spherical harmonics $Y(l,m)$)

3/31/2014

$$H_{ab} = \langle a | \mathcal{H} | b \rangle$$

$$\vec{r} = r\left(Y_{1,-1}\,\frac{\hat{x}+i\hat{y}}{\sqrt{2}} + Y_{1,1}\,\frac{\hat{x}-i\hat{y}}{\sqrt{2}} + Y_{1,0}\,\hat{z}\right)$$

$$xz = \left(T_{2,-1} - T_{2,1}\right)/4$$

$$= \frac{\langle n J' \| T_\ell \| n' J' \rangle}{\sqrt{2J+1}} \langle J' \ell, M' m \mid JM \rangle$$

reduced matrix element — electric Gordon coeff.

$M = M' + m$

triangle rule

$$|J' - J| \leq \ell \leq |J' + J|$$

Wigner-Eckart theorem

$E1, M1$

$\langle a | \vec{r} | b \rangle$ $\langle a | \vec{L} | b \rangle$ vectors $Y_{1,m}$

dipole selection rules $|\Delta J| = 0, 1$

$\Delta m = 0, \pm 1$

E_1 opposite
M_1 same
} parity

$E2$ $\times 2$ $Y_{2,m}$

Selection rules for quadrupole trans.

$|\Delta J| = 0, 1, 2$

$\Delta m = 0, \pm 1, \pm 2$

Polarization of light — E1

$\hat{\varepsilon} \langle b | \vec{r} | a \rangle$ — projects out $r_{1,\pm 1}$

$\Delta m = \pm 1$ — σ_+, σ_-

$\hat{x} \pm i\hat{y}$ / $\sqrt{2}$

Circular

Linear pol.: $\hat{x}$ or $\hat{y}$

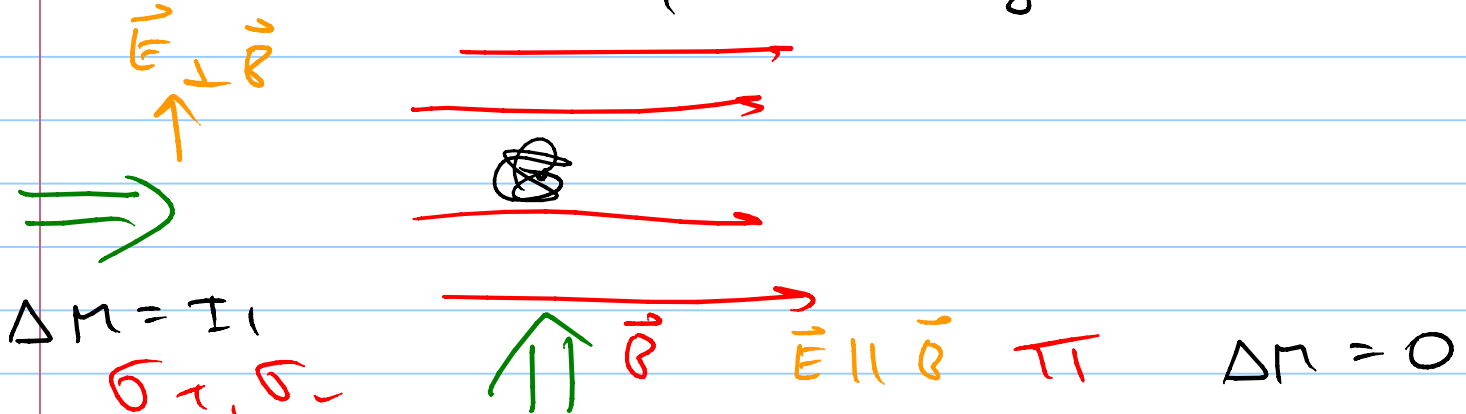
linear superposition of σ_+, σ_-

$\hat{k}$, propagation along $\hat{z}$

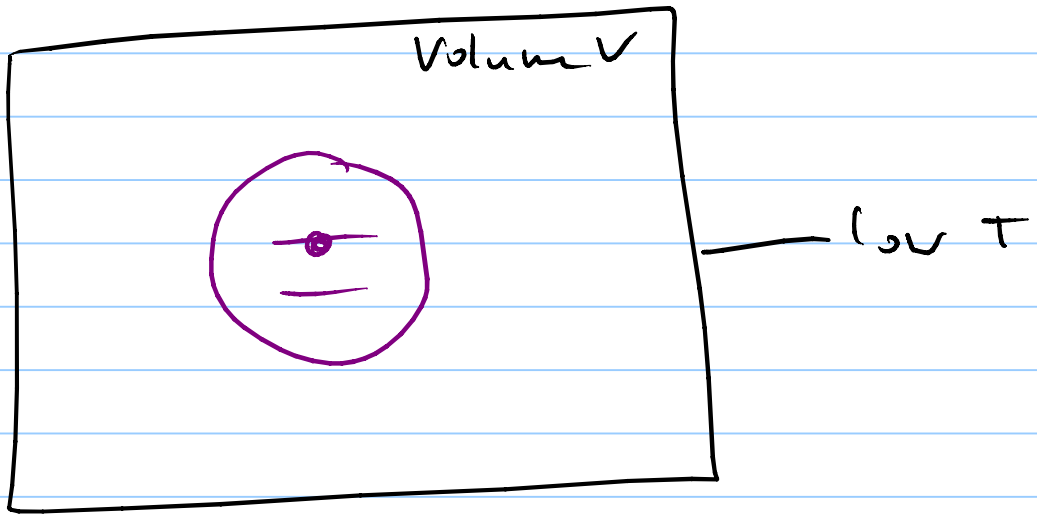
$\hat{z}$, quant. axis $\parallel \vec{E} \Rightarrow \vec{E} \parallel \vec{r}$

$z = r Y_{1,0}$

$\Delta m = 0$, π light



Atom-light interaction



atom will irreversibly decay to ground state, lifetime τ

$\Gamma = \tau^{-1}$ ~~////////~~ e

$2\,eV \hat{=} 20{,}000\,K$

$\hat{=} 25\,meV$

$kT_{@\,room\,temp}$

$kT \ll \hbar\omega_{eg}$

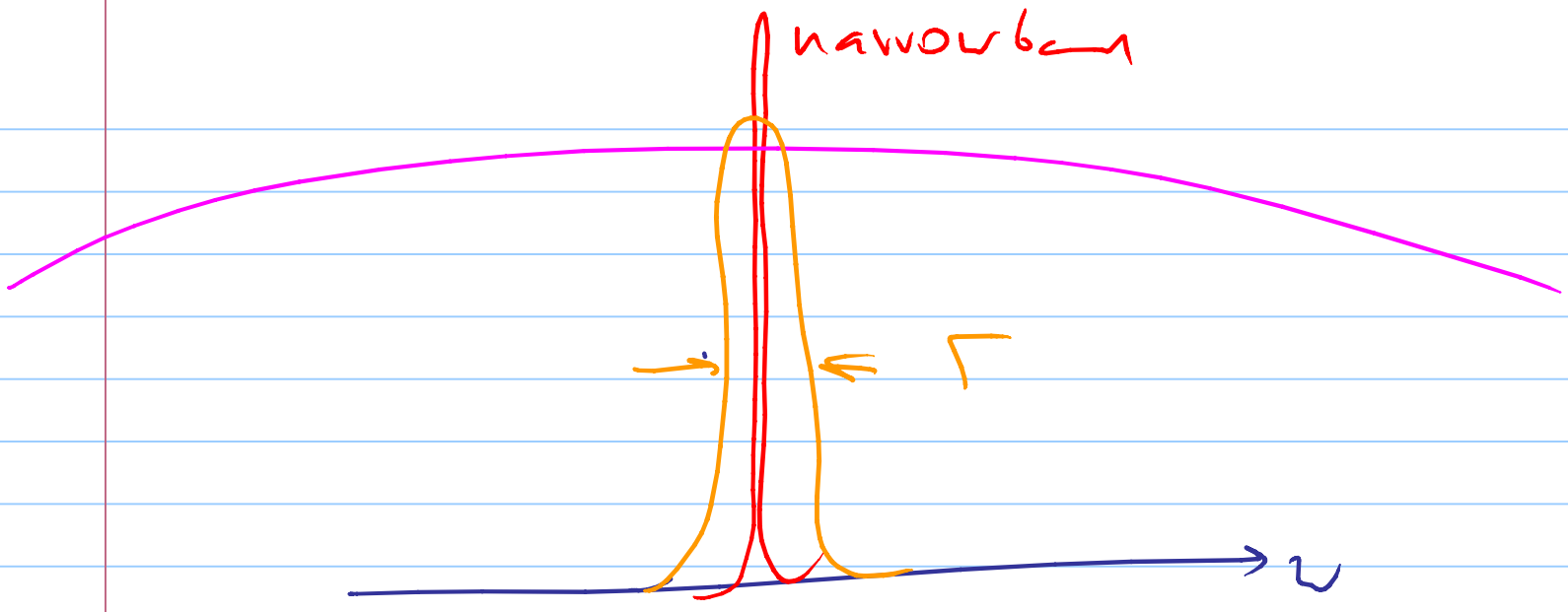
g

e

Now: add light with a bandwidth $\Delta\omega$

$\Delta\omega \ll \Gamma$ narrowband exc. $\Delta\omega \to 0$

$\Delta\omega \gg \Gamma$ broadband radiation



Broadband

Spectral density $S(\omega) = \frac{(\text{energy})}{(\text{volume})(\text{freq. int.})}$

or intensity per unit freq. int.

$I(\omega) = c\,S(\omega) = \frac{(\text{energy})}{[\text{area}][\text{time}][\text{freq. int.}]}$

Monochromatic radiation

ω, $\mathcal{E}$ (el. field amplitude)

or Intensity I

look at $t \ll \Gamma^{-1}$

Strong driving field, Rabi flopping

$|g\rangle$, $|e\rangle$

$$a_e(t) = \frac{1}{i\hbar}\int_0^t V_{eg}\, e^{-i(\omega - \omega_{eg})t'}\, dt'$$

$V_{eg}\, e^{-i\omega t}$

$$= \frac{V_{eg}}{i\hbar}\left[\frac{e^{-i(\omega-\omega_{eg})t}-1}{\omega-\omega_{eg}}\right]$$

$$P_e = |a_e|^2 = \frac{|V_{eg}|^2}{\hbar^2}\frac{\sin^2[(\omega-\omega_{eg})t/2]}{[(\omega-\omega_{eg})/2]^2}$$

For short times:

$$P_e = \frac{|V_{eg}|^2}{\hbar^2}t^2 \qquad \propto t^2 \qquad \text{coherent process}$$

Longer times:

Rabi oscillations

here deriving perturbatively

(i.e. $|a_e|^2 \ll 1$ [off resonant case])

Let's do it now exactly.

$$H_{eg} = \langle e|H|g\rangle = -\underbrace{\vec{d}\cdot\vec{E}_0}_{\hbar\omega_R}\cos\omega t$$

to compare with spin 1/2:

shift the ground state to $-\frac{1}{2}\hbar\omega_{eg}$

$$H = \frac{1}{2}\hbar\omega_{eg}\underbrace{(|e\rangle\langle e| - |g\rangle\langle g|)}_{\sigma_z} + \hbar\omega_R\underbrace{(|g\rangle\langle e| + |e\rangle\langle g|)}_{\sigma_x}\cos\omega t$$

$$= \frac{1}{2}\hbar\omega_{eg}\hat{\sigma}_z + \hbar\omega_R \cos\omega t\,\hat{\sigma}_x$$

H corresponds to spin $1/2$ in a static z-field and a linearly polarized oscillating field along $\hat{x}$.

$$H = \frac{1}{2}\hbar\omega_{eg}\hat{\sigma}_z + \frac{1}{2}\hbar\omega_R\left(\hat{\sigma}_x\cos\omega t + \hat{\sigma}_y\sin\omega t\right) + \frac{1}{2}\hbar\omega_R\left(\hat{\sigma}_x\cos\omega t - \hat{\sigma}_y\sin\omega t\right)$$

$\Rightarrow$ co counter rotating magnetic Fields

Counter: rotates at $\omega_{eg}+\omega$ in the frame rotating at the Larmor freq. ω_{eg}

If $|\omega-\omega_{eg}| \ll \omega_{eg}$ and $\omega_R \ll \omega_{eg}$

we can neglect the last term

Rotating Wave Approx.

Our results for spin $1/2$

$$P_e(t) = \frac{\omega_R^2}{\Omega_R^2}\sin^2\left(\frac{1}{2}\Omega_R t\right) \quad \text{if } P_e(0)=0$$

$$\Omega_R = \sqrt{\omega_R^2+\Delta^2}$$ generalized Rabi Freq.

$$\Delta = \omega-\omega_{eg}$$ detuning

Broadband excitation

$$|a_e(t)|^2 = \left(\frac{\omega_R}{2}\right)^2 \frac{\sin^2\left(\frac{\Delta}{2}t\right)}{(\Delta/2)^2}$$

Broadband

$W(\omega)$ energy density

$$\frac{1}{2}\varepsilon_0 E_0^2 = \int d\omega\, W(\omega)$$

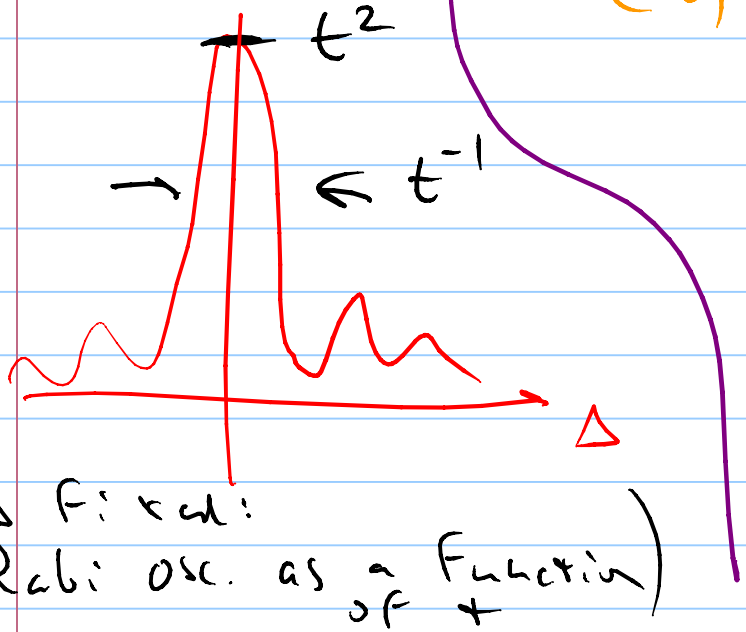
$$\omega_R^2 = \frac{1}{\hbar^2}\left(d_x E_0\right)^2 \rightarrow \frac{2}{\varepsilon_0}\frac{d_x^2}{\hbar^2}\int d\omega\, W(\omega)$$

$$P_e(t) = \frac{d_x^2}{2\varepsilon_0\hbar^2}\int d\Delta\, W(\omega_0+\Delta)\,\frac{\sin^2\left(\frac{\Delta}{2}t\right)}{(\Delta/2)^2}$$

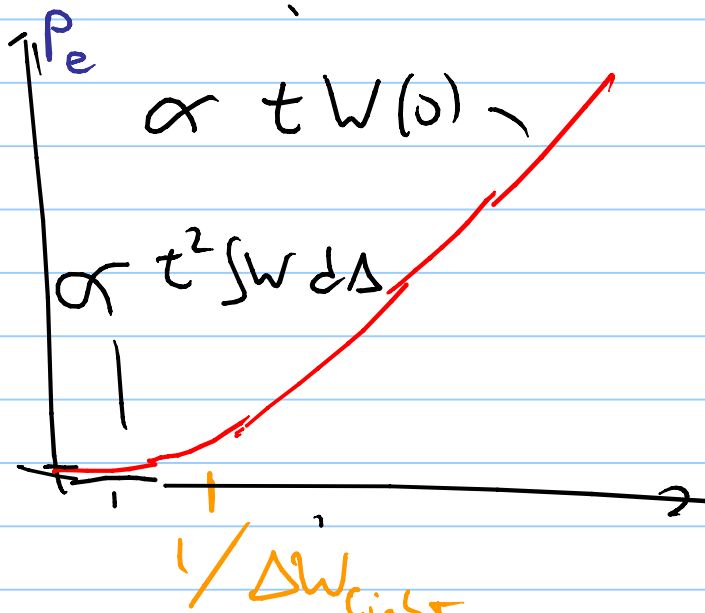
$W(\omega)$ if function is flat

$F(t,\Delta)$

$F(t,\Delta)$



(Δ Fixed: Rabi osc. as a Function of t)



4/2/2014

$$F(t,0) = t^2$$

$$F(t,\Delta) \xrightarrow[\text{large } t]{} 2\pi t\,\delta(\Delta)$$

For large t:

$$\text{Rate } R = \frac{|a_e(t)|^2}{t} = \frac{d_x^2}{2\varepsilon_0 t^2} 2\pi \langle W(\omega_{eg}) \rangle$$

$$\vec{d}^2 = d_x^2 + d_y^2 + d_z^2$$

$$\Rightarrow d_x^2 = \frac{1}{3} d^2$$

$$R = B_{ge} \langle W \rangle$$

$$B = \frac{\pi d^2}{3\varepsilon_0 t^2}$$

Fermi's Golden Rule:

$$R = 2\pi \frac{|t_{ba}|^2}{t^2} \delta(\omega - \omega_{ba})$$

↳ needs integration

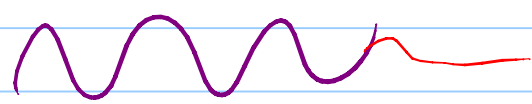
- spectrum of ext. fields
- continuum of final states

Rabi resonance	Rate equation
single final state	many final states OR
single mode ω	many modes ω
discrete	continuum
reversible	irreversible *
unitary	dissipative *

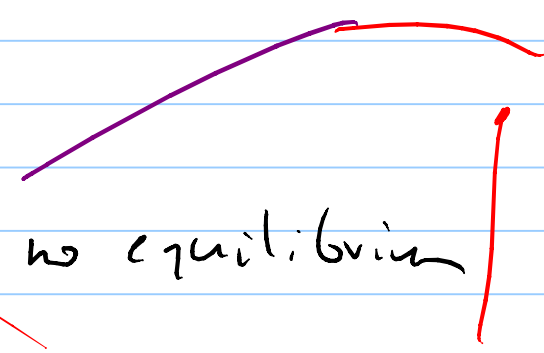
* due to integration over ~~∞~~ number of modes/states

So far:

- $\langle a | \mathcal{H} | b \rangle$
- S. eq., pert. theory



undamped



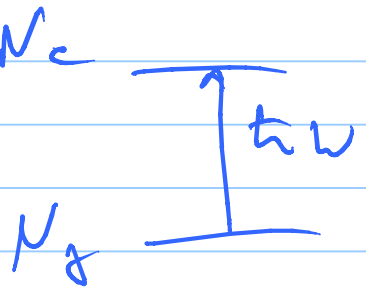
no equilibrium

- One missing element: spont. em
  - A & B Einstein
  - Quant. of em field

• microscopic calculation of A

Einstein's A and B coefficients

$$\frac{N_e}{N_g} = \frac{g_e}{g_g} \exp\left(\frac{-\hbar\omega}{kT}\right)$$



Spectral density in blackbody cavity

$$W(\omega)\, d\omega = \underbrace{\langle n(\omega)\rangle}_{\text{photon \#}} \hbar\omega \; \underline{\rho(\omega)}\, d\omega$$

$$\frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \qquad \frac{\omega^2}{\pi^2 c^3}$$

$$= \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}$$

Rate eq. for atoms:

$$\dot{N}_e = -\Big[\underbrace{W(\omega)\, B_{eg}}_{\text{stim em}} + \underbrace{A_{eg}}_{\text{spont. em}}\Big] N_e$$

$$+ \underbrace{W(\omega)\, B_{ge}}_{\text{Abs}} N_g$$

equilibrium $\dot{N}_e = \dot{N}_g = 0$

$$\Rightarrow W(\omega) = \frac{A_{eg} N_e}{N_g B_{ge} - N_e B_{eg}} \Rightarrow$$

$$= \frac{A_{eg}}{\frac{g_g}{g_e} e^{\hbar\omega/kT} B_{ge} - B_{eg}}$$

$$A_{eg} = B_{eg} \frac{\hbar\omega^3}{\pi^2 c^3} \quad (a)$$

$$g_g B_{ge} = g_e B_{eg}$$

(a) $\bar{n}$ $\qquad \bar{n} A_{eg} = \underbrace{W(\omega) B_{eg}}_{\text{Stim. em}}$

$$\frac{g_e}{g_g} \bar{n} A_{eg} = \underbrace{W(\omega) B_{ge}}_{\text{Abs}}$$

Single state, $g_e = g_g = 1$

$$\frac{\text{rate for em.}}{\text{rate for abs}} = \frac{\bar{n}+1}{\bar{n}}$$

Next: We will show that $\frac{n+1}{n}$

is valid not only for an average over many modes, but for each single mode

Quantization of the radiation field

Single mode of the em field

$$\vec{A}(r,t) = \hat{e}\, A\, e^{i(\vec{k}\vec{r} - \omega t)} + c.c$$

Pol.

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

total energy (el. & magnetic)

$$U = \frac{\varepsilon_0}{2}\omega^2 |A|^2 V$$

Analogous to HO

Define Q, P

$$A^* = \sqrt{\frac{1}{\varepsilon_0 V}}\frac{1}{\omega}(\omega Q \pm iP)$$

Vol

$$U = \frac{1}{2}\left(\omega^2 Q^2 + P^2\right)$$

Postulate as quantum theory:

$$[Q, P] = i\hbar$$

intuition

a, a^+

$$a^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega Q - iP)$$

$$[a, a^+] = 1$$

$$H \to H = \hbar\omega\left[a^+ a + \frac{1}{2}\right]$$

$a^+ a$: # of quanta

non-vanishing matrix elements

$$\langle n-1 | a | n \rangle = \sqrt{n}$$

$$\langle n+1 | a^+ | n \rangle = \sqrt{n+1}$$

$A \to Q, P \to a, a^+$

H

$$E = -\frac{\partial A}{\partial t}$$

$$E = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left[a\,\hat{e}\,e^{i(kz-\omega t)} - c.c.\right]$$

a $\qquad$ a^+

Schr. picture

$$H_{INT} = -\mathcal{d}E = e\vec{r}E$$

$$= ie\sqrt{\frac{E\omega}{2\varepsilon_0 V}}\,\vec{r}\left[a\,\hat{e}\,e^{-i\omega t} - c.c\right]$$

$$\sqrt{\frac{E\omega}{2\varepsilon_0 V}} = \varepsilon_1$$

Matrix elements of H_{INT}

(For single mode radiation field)

$$I = |a\rangle|n\rangle \qquad F\,|b\rangle|n'\rangle$$

$$\langle F|H_{INT}|I\rangle = ie\varepsilon_1 \vec{r}_{a,b}\,\hat{e}\left[\sqrt{n}\,\delta_{n',n+1} + \sqrt{n+1}\,\delta_{n',n+1}\right]$$

ΔE_s ; $E = n$

Rates of ΔE_s and E_n

$$\sum_{n'}\left|\langle b,n'|H_{INT}|a,n\rangle\right|^2 = (\;)\begin{cases} n+1 & E_n \\ n & \Delta E_s \end{cases}$$

Spontaneous emission

Microscopic calculation of A coeff.

Fermi's Golden Rule $\frac{\omega^2 V}{8\pi^3 c^3}$

$$\frac{dA}{d\Omega} = R = 2\pi \frac{|H_{GE}|^2}{\hbar^2} \frac{d\rho(\omega)}{d\Omega}$$

ρ density of states (per pol., per unit freq interval

$$E = |e\rangle \otimes |0\rangle_1 |0\rangle_2 |0\rangle_3$$

Modes

$$G = |g\rangle \otimes |0\rangle_1 |1\rangle_2 |0\rangle_3$$

$$|H_{GE}|^2 = e^2 \varepsilon_1^2 (\vec{r}_{gc} \cdot \hat{e})^2 (\cancel{n}^{\,0} + 1)$$

$$= e^2 \frac{\hbar\omega}{2\varepsilon_0 V} (\vec{r}_{gc}\, \hat{e})^2$$

$$\frac{dA}{d\Omega} = \frac{2\pi}{\hbar^2} \frac{\hbar\omega_{eg}}{2\varepsilon_0 \cancel{V}} (\vec{d}\hat{e}) \frac{\omega_{eg}^2 \cancel{V}}{8\pi^3 c^3} = \frac{(\vec{d}\cdot\hat{e})^2 \omega_{eg}^3}{2\pi \varepsilon_0 \hbar c^3} \frac{1}{4\pi}$$

$$A = \int (\quad) \left(\vec{I}\cdot\hat{e}\right)^2 g(\omega)\,\frac{d\Omega}{4\pi} =$$

pol. 1 $\quad \vec{I}\cdot\hat{e} = \sin\theta$

pol 2 $\quad 0$

$$|\vec{d}|^2 \underbrace{\int \sin^2\theta\,\frac{d\Omega}{4\pi}}_{2/3}$$

$$A = \frac{4\,\vec{d}^{\,2}}{3\,(4\pi\varepsilon_0)}\,\frac{\omega_{eg}^2}{\hbar c^3}$$

4/7/2014

MIDTERM on Friday, 4/18 (in 56-114)
ONE MORE FRIDAY CLASS 4/25 (in 56-154) instead of Monday 4/28

in atomic units

$$\left.\begin{aligned} \omega &\approx 1 \\ r_{eg} &\approx 1 \\ c &= \frac{1}{\alpha} \end{aligned}\right\} \quad A \approx \alpha^3 = 3\cdot 10^{-7}$$

$\frac{A}{\nu_{eg}}$ is small

$$3 \times 10^{-7} \cdot 2 R_y d \simeq 10^9 / s$$

$$\tau = 1/A \simeq 1 \text{ nsec}$$

often 10 , - 100 ns because

$$\omega_{eg} < \frac{2 R_y d}{\hbar}$$

visible

ω^3 dependence

radiative lifetime for μwave transition

is 10^{15} longer $\rightarrow$ months

magnetic dipole (instead of electric dipole)

α^2 weaker $\rightarrow$ 1000 years lifetime

Clicker question

- Can an E2 transition (quadrupole) driven by a plane wave?

  A yes ✓

  B no

  $e^{ikr} \sim 1 + ikr$

- Can spont. em be described as stimulated emission by the zero-point field?

  A yes

  B no

  $E = \left(n + \frac{1}{2}\right)\hbar\omega$

  Rate $(n+1)$

- Spont. em $\propto \omega^3$

  What is the dependence in 1D

  A ω^3

  B ω^2

  C ω ✓

- Where does the other power of ω come from?

A Atomic matrix element

B Dipole approx

C Field quant ✓

$\hbar\omega = 2\varepsilon_0 V E_1^2$

RVA

For electronic transitions, do we always get the counter-rotating term?

A Yes

B No ✓ ← can be zero due to angular mom. selection rules

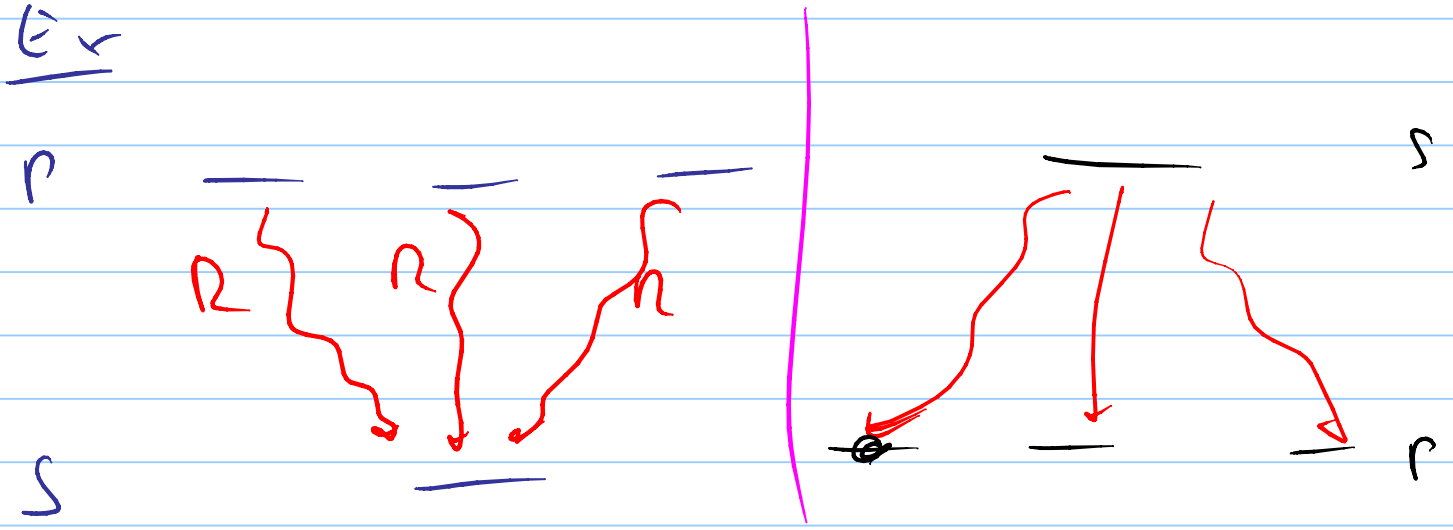
Lamb shift: Is it due to counter-rotating term

A Yes ✓ is the AC stark effect of the vacuum fluctuations

B No

Note on degeneracy factors

$$A \propto B_{eg} \qquad B_{ge} = \frac{g_e}{g_g} B_{eg}$$



Abs: $B_{sp} \propto 3R$ Abs: $B_{ps} \propto R$

$\Gamma_{\text{stim Em}} B_{ps} \propto R$ $\Gamma_{\text{stim Em}} B_{sp} \propto 3R$

Some def. of line strength

$$S \propto \sum_{m_s, m_s} R(m_I, m_F)$$

then $A \propto \frac{S}{g_e}$

Note: Our derivation of the A coeff assumed one final state

$\bar{d} = d\,\hat{e}_z \Rightarrow P, m=0 \longrightarrow S$

Fully quantized Hamiltonian

Semiclassical $\mathcal{H}$

$$H = \frac{1}{2}\omega_{eg}\hat{\sigma}_z + \underbrace{\hbar\omega_R}_{\vec{d}\,\vec{E}_0}\cos\omega t\ \sigma_x$$

$$\sigma_x \to |g\rangle\langle e| + |e\rangle\langle g|$$

$$|g\rangle\langle e| = \hat{\sigma}_-, \quad |e\rangle\langle g| = \hat{\sigma}_+$$

$$\vec{E}(0,t) \to \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,\hat{e}\,(\hat{a} + \hat{a}^\dagger)$$

Schrodinger picture $i\hat{a} - i\hat{a}$ shift the origin of time

Single photon Rabi freq:

$$-\vec{d}\cdot\hat{e}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} = \frac{\hbar\omega_1}{2}$$

$$H = \frac{1}{2}\hbar\omega_{eg}\hat{\sigma}_z - \frac{1}{2}\hbar\omega_1(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) + \hbar\omega\,\hat{a}^\dagger\hat{a}$$

$$\hat{\sigma}_+|g\rangle = |e\rangle$$

$$\hat{\sigma}_-|e\rangle = |g\rangle$$

Hilbert space: atom $\otimes$ light

i.e. states $(|e\rangle$ or $|g\rangle) \otimes |n\rangle$

Interaction has four terms

$\hat{\sigma}_+ \hat{a}$ Abs $|n\rangle \to |n-1\rangle$, $|g\rangle \to |e\rangle$

$\hat{\sigma}_- \hat{a}^\dagger$ Em $|n\rangle \to |n+1\rangle$, $|e\rangle \to |g\rangle$

} energy conserving terms

$\hat{\sigma}_+ \hat{a}^\dagger$ atom exc + photon em $|g\rangle \to |e\rangle$

$\hat{\sigma}_- \hat{a}$ $|e\rangle \to |g\rangle$ + photon absorption atom de-exc.

} "off shell"

→ no transitions, but shifts

Lamb shifts (Bloch-Siegert shifts)

Note:

$|n\rangle = |2\rangle \quad |g\rangle$

only non-vanishing term is

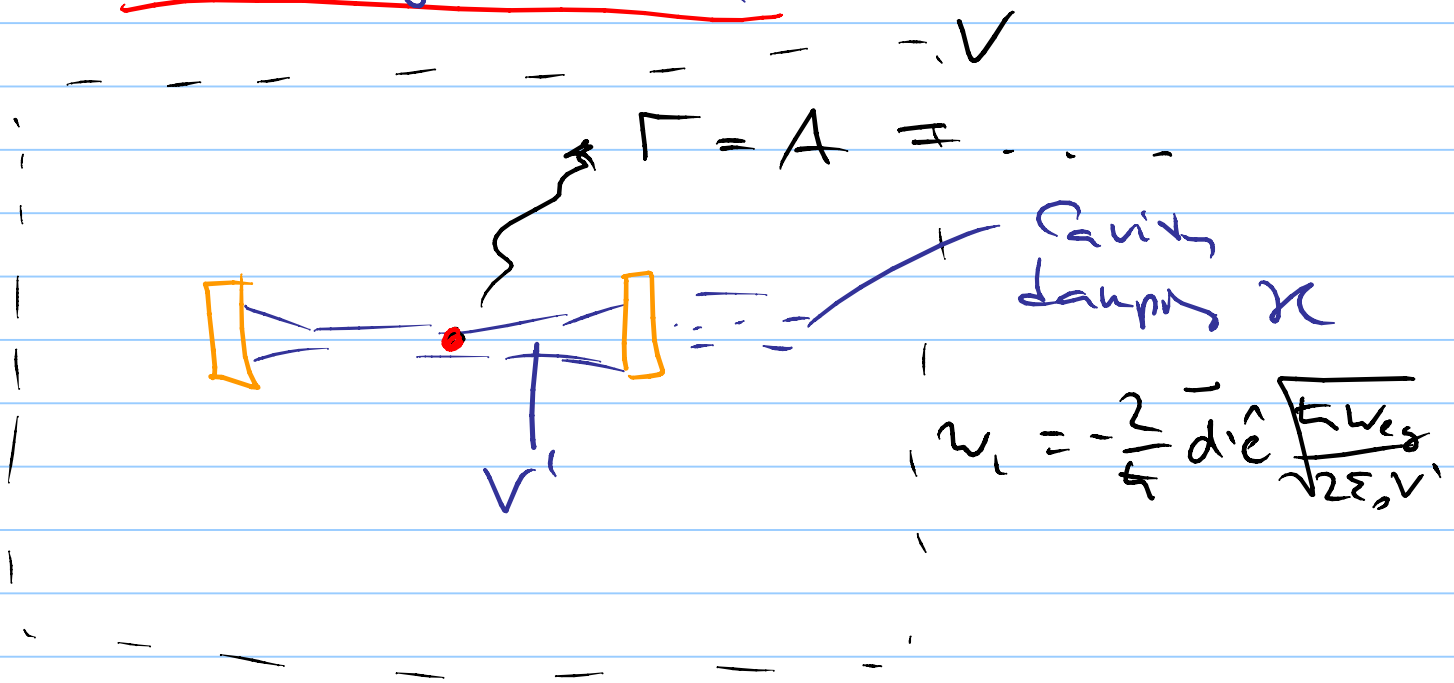
$\hat{\sigma}_+ \hat{a}^\dagger$

→ Lamb shift

Fully quantized Hamiltonian in the RWA

$$\hat{H}_{RWA} = \frac{1}{2}\hbar\omega_{eg}\hat{\sigma}_z - \frac{1}{2}\hbar\omega_1(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger) + \hbar\omega\,\hat{a}^\dagger\hat{a}$$

Vacuum Rabi oscillation, Jaynes-Cummings model



Small V' $\Rightarrow$ strong coupling regime $\omega_1 \gg \Gamma, \kappa$ of cavity QED

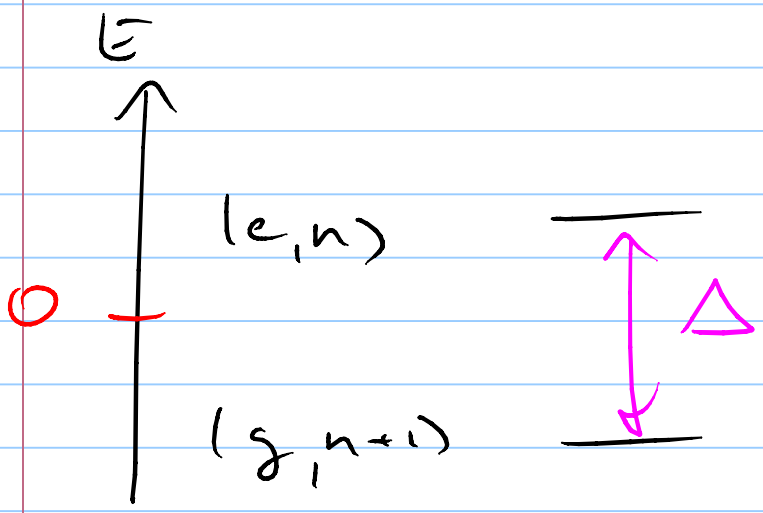
$\hat{H}_{RWA}$ couples only pairs of states

$\hat{\sigma}_+\hat{a}$

$|e\rangle \otimes |n\rangle$ $\qquad$ $|g\rangle \otimes |n+1\rangle$

$\hat{\sigma}_-\hat{a}^\dagger$

$$\Delta = \omega - \omega_{eg}$$



shift origin

$$H_{RWA,n} = \frac{\hbar}{2}\begin{pmatrix} \Delta & -\omega_1\sqrt{n+1} \\ -\omega_1\sqrt{n+1} & -\Delta \end{pmatrix}$$

4/9/2014

Identical to spin 1/2 in B fields with a static x field B_x of $2\omega_1\sqrt{n+1}$

On resonance
$\Delta = 0$

$n = 0$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -\omega_1 \\ -\omega_1 & 0 \end{pmatrix}$$

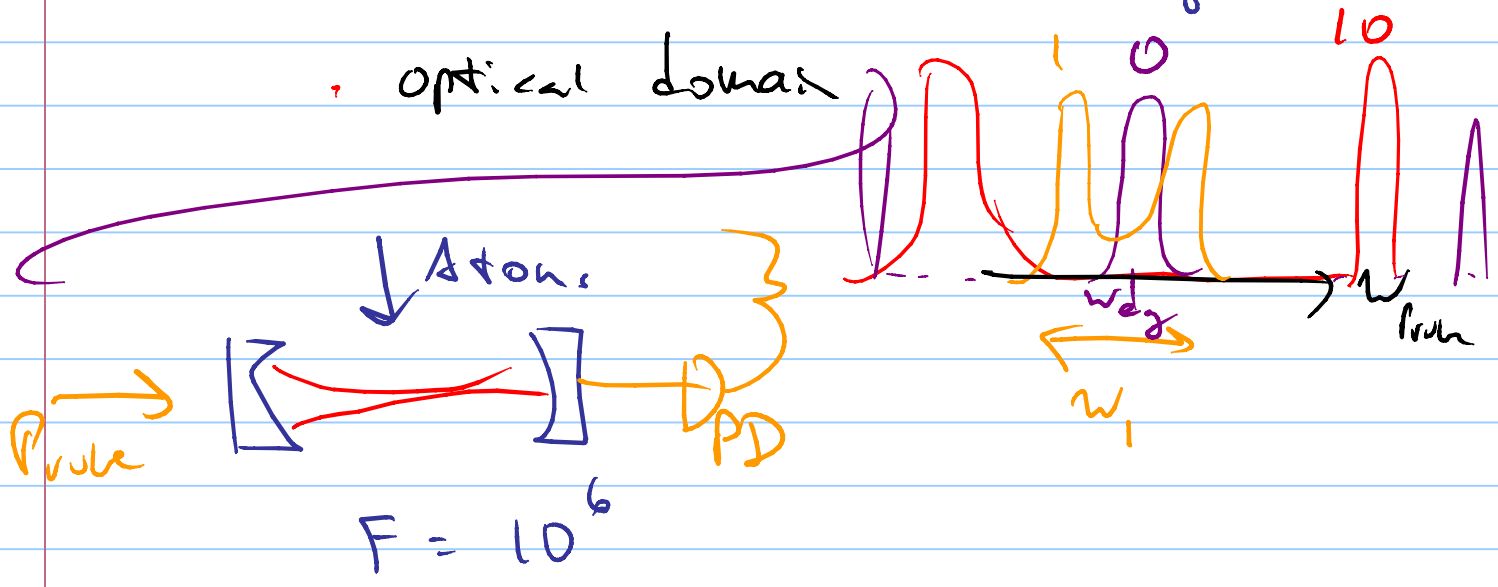
$|I\rangle = |e,0\rangle \xrightarrow[\text{oscillation}]{} |g,1\rangle$

Vacuum Rabi osc $P_e = \frac{1}{2}(1+\cos\omega_1 t)$

Periodic spont. em & reabsorption of the same photon

Expt: μ wave Rydberg atoms + Superconducting cavities

- optical domain



$F = 10^6$

Observation of Normal-Mode Splitting for an Atom in an Optical Cavity

R. J. Thompson, G. Rempe, and H. J. Kimble
Norman Bridge Laboratory of Physics 12-33, California Institute of Technology, Pasadena, California 91125
(Received 4 November 1991)

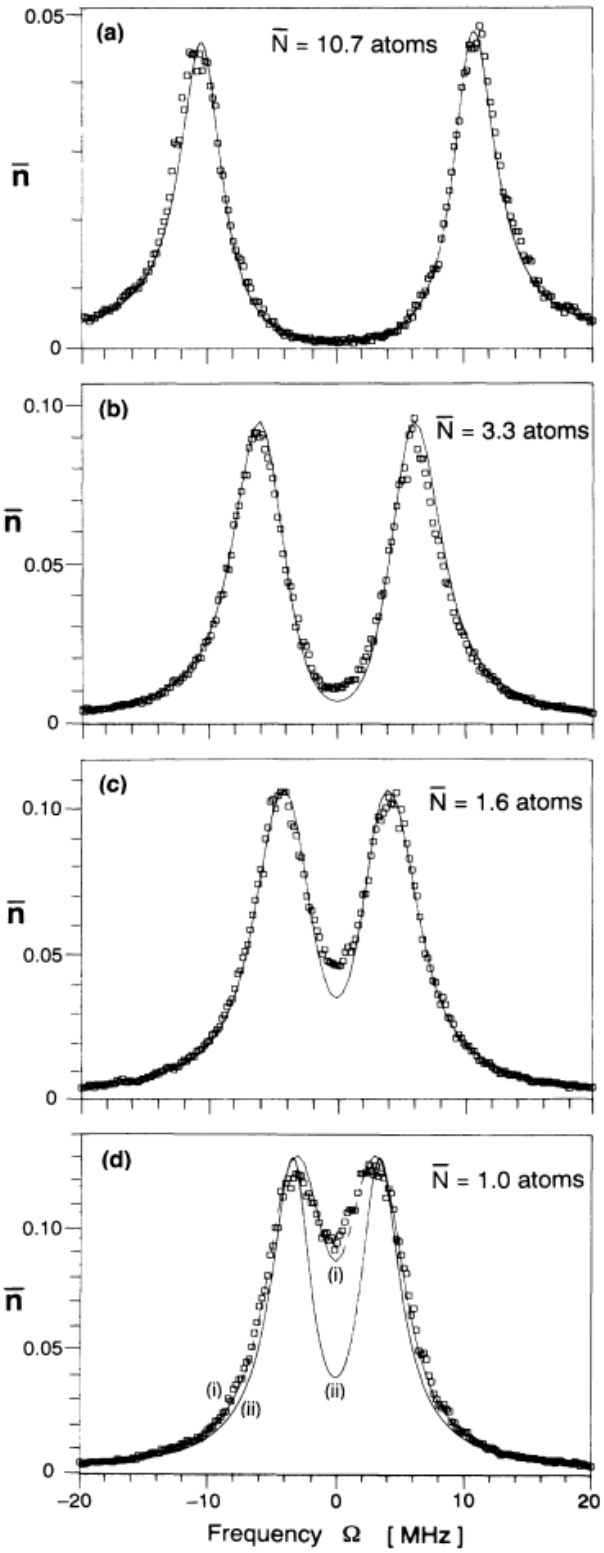


FIG. 2. Intracavity photon number $\bar{n}$ vs probe frequency Ω for four values of $\bar{N}$ and with $\omega_C \simeq \omega_A$. Curves in (a)–(c) and curve i in (d) are theoretical fits to the data including fluctuations in atomic number and position. Curve ii in (d) is from Eq. (1) for a single intracavity atom with optimum coupling g_0.

Single photon Rabi flopping has been observed

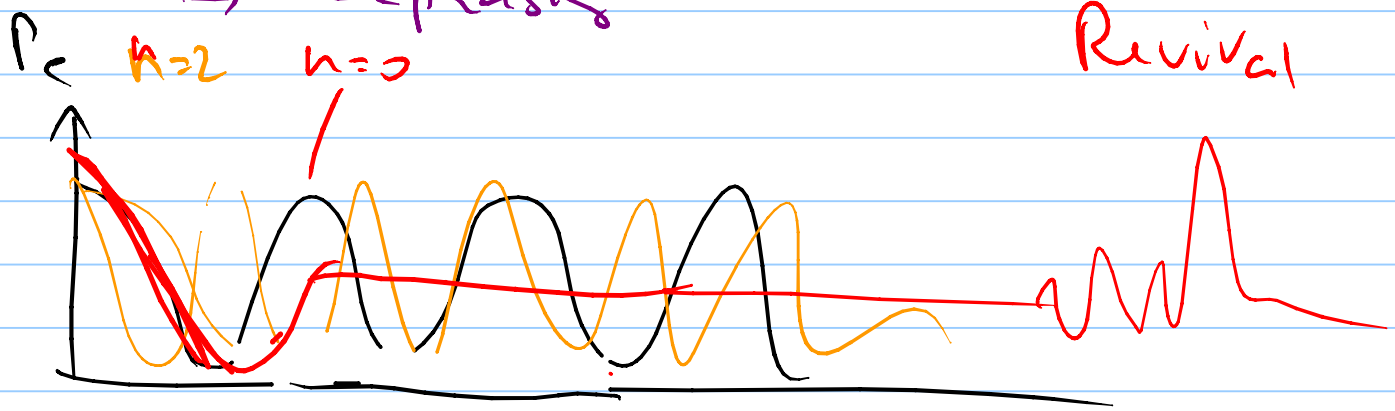
$|e,0\rangle \leftrightarrow |g,1\rangle$

Now: initial photon state is not $|0\rangle$, but a thermal state or coherent state ⇒ superposition

$\Omega_n = \omega_1 \sqrt{n+1}$

Rabi osc have different freq for different doublets n

→ dephasing

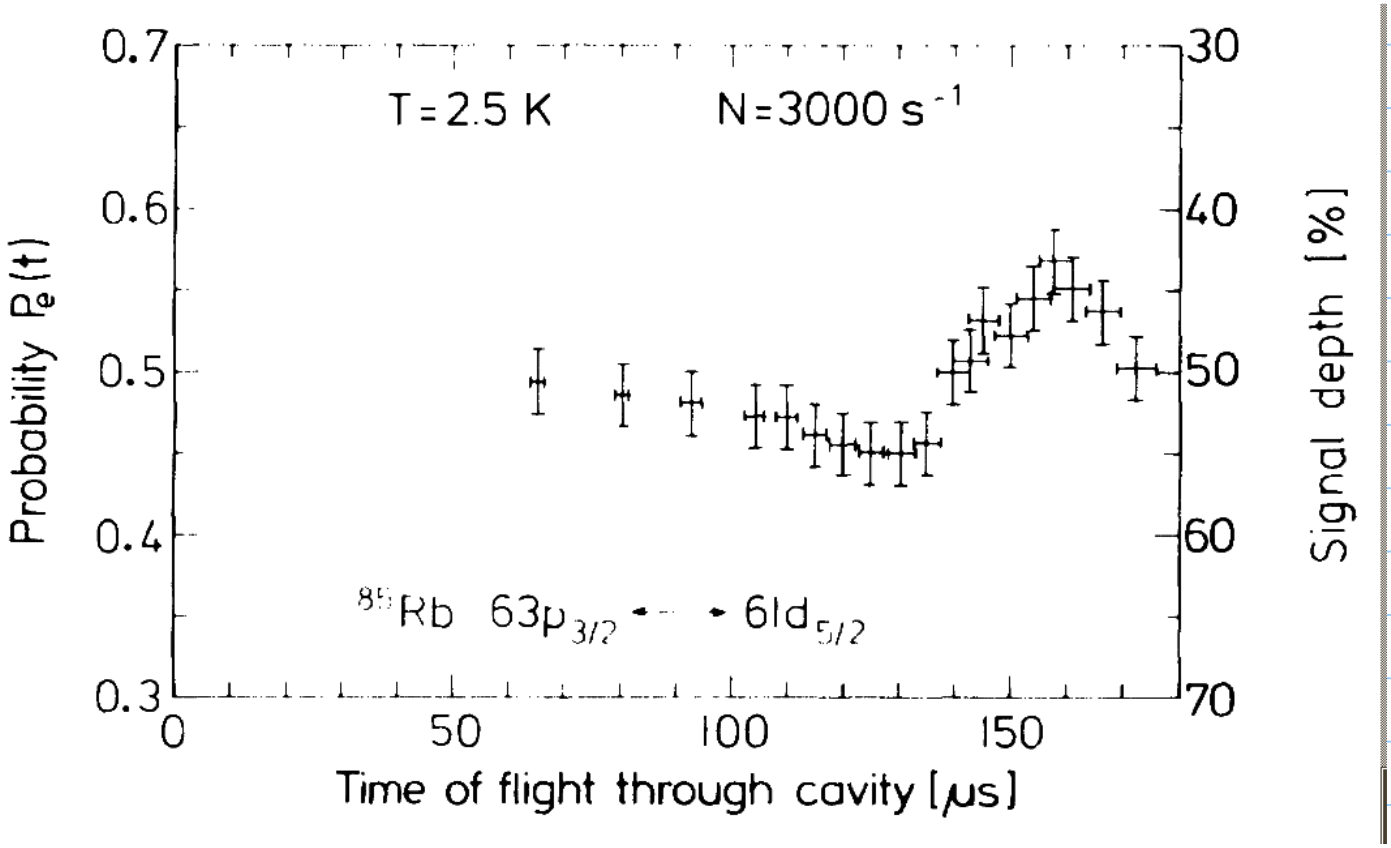


Observation of Quantum Collapse and Revival in a One-Atom Maser

Gerhard Rempe and Herbert Walther
Sektion Physik, Universität München and Max-Planck-Institut für Quantenoptik, D 8046 Garching bei München, Federal Republic of Germany

and

Norbert Klein
Fachbereich Physik, Universität Wuppertal, D 5600 Wuppertal, Federal Republic of Germany
(Received 27 October 1986)



Coherent state

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \sum a_n |n\rangle$$

$|e\rangle \otimes |\alpha\rangle$ ($t=0$) $\xrightarrow[t]{\text{unitary, no "spontaneity"}}$

$$\sum a_n \left(\cos\frac{\Omega_n t}{2} |e\rangle \otimes |n\rangle + i \frac{\sin \Omega_n t}{2} |g\rangle \otimes |n+1\rangle \right)$$

$\langle n \rangle \gg 1 \qquad \frac{\Delta n}{\langle n \rangle} \ll 1$

$\Rightarrow$ Semiclassical Rabi flopping

with $\Omega_R = \Omega_n = \sqrt{n}\,\Omega_1$

$\sqrt{n} \approx \sqrt{n+1}$

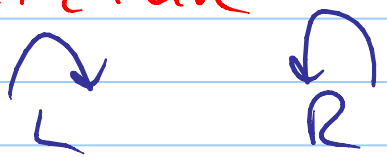
RWA — revisited

- $H = H_0 - \frac{1}{2}\hbar\Omega_1\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^+\right)$

  $\rightarrow$ 4 terms

- Semiclassical picture

  Consider



We quantize along $\hat{k}$

$\vec{E} \,|\, \vec{B} = B_0\,\hat{\imath}\cos\omega t = \vec{B}_R + \vec{B}_L$

$\vec{B}_R = \frac{B_0}{2}\left(\hat{\imath}\cos\omega t + \hat{\jmath}\sin\omega t\right)$

$\vec{B}_L$

$\vec{B}_R \leftrightarrow \vec{B}_L \qquad \omega \leftrightarrow -\omega$

$$\vec{B}_R = \frac{B_0}{2}\left[\hat{i}\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)+\hat{j}\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)\right]$$

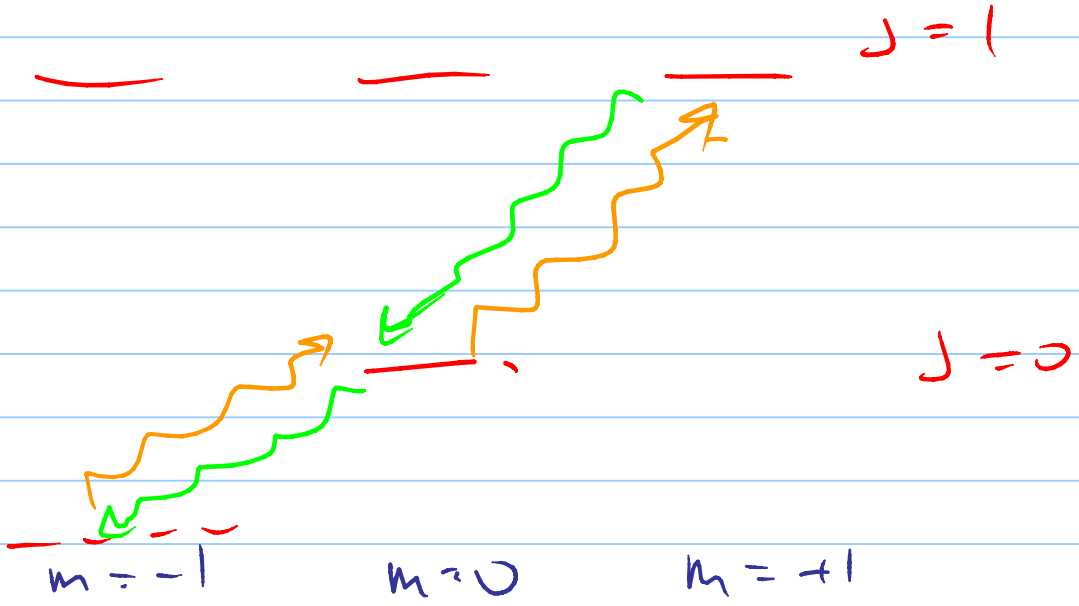
$$= \frac{B_0}{4}\left[(\hat{i}-i\hat{j})\,e^{i\omega t} + (\hat{i}+i\hat{j})\,e^{-i\omega t}\right]$$

this picks out the x + iy tensor component $\Delta m = +1$

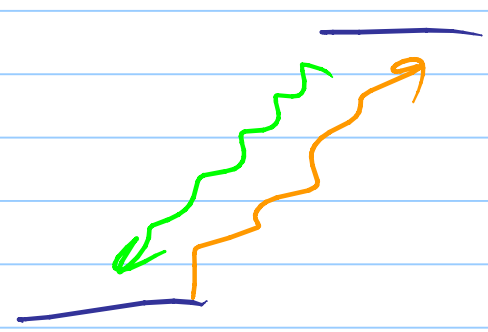
Abs.

$\Delta m = -1$

Stim em

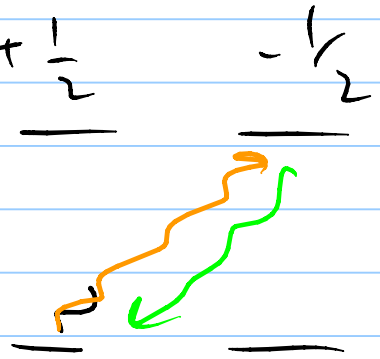


2 levels $m = +1$
$m = 0$



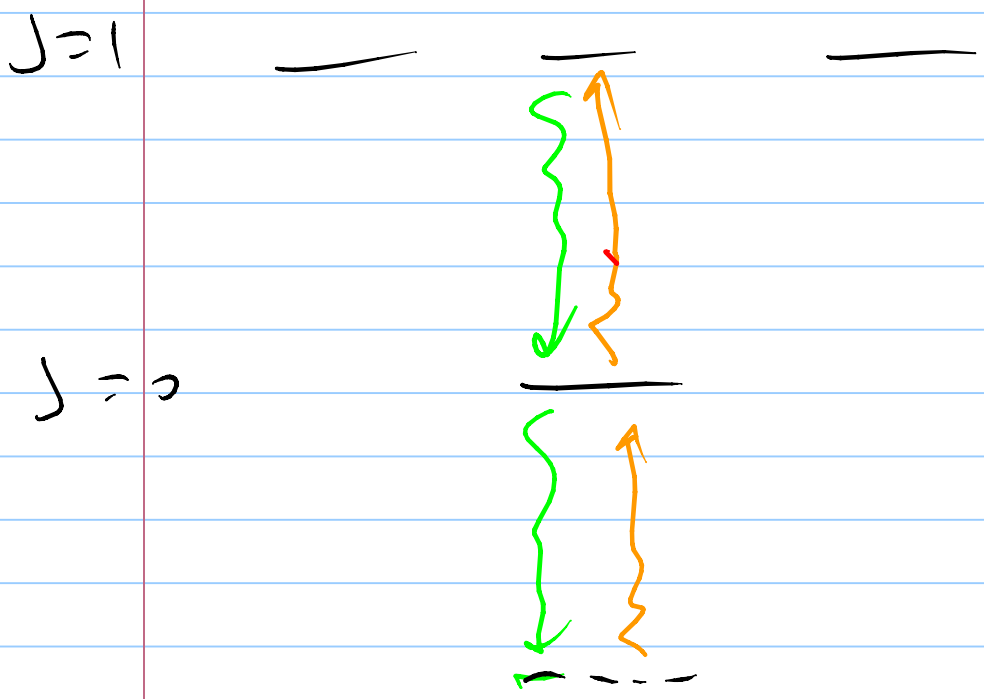
RWA is exact

$^2S \leftrightarrow {}^2S$



Let's quantize along $\hat{x}$ π light

$$\vec{\beta} = \frac{\beta_0}{2}\hat{\epsilon}\left(e^{i\omega t} + e^{-i\omega t}\right) \qquad \Delta m = 0$$



Counterrotating terms can be zero for σ_+, σ_- light due to angular momentum selection rules.

4/14/2014

Why is $e^{\pm i\omega t}$ abs/em.

$$\dot{a}_1 = (\ )\langle 1|V|2\rangle a_2$$

$\langle 1|V|2\rangle$ ← Drive

$$\Delta a_1 = (\ )\int_t^{t+\Delta t} \langle 1|V|2\rangle \, dt$$

$e^{i\omega_1 t}$ | $e^{-i\omega_2 t}$

$\sim e^{i\omega t}$

$$\int e^{i(\omega_1 - \omega_2 + \omega)t} \, dt$$

$\neq 0$ for $\omega = -(\omega_1 - \omega_2)$

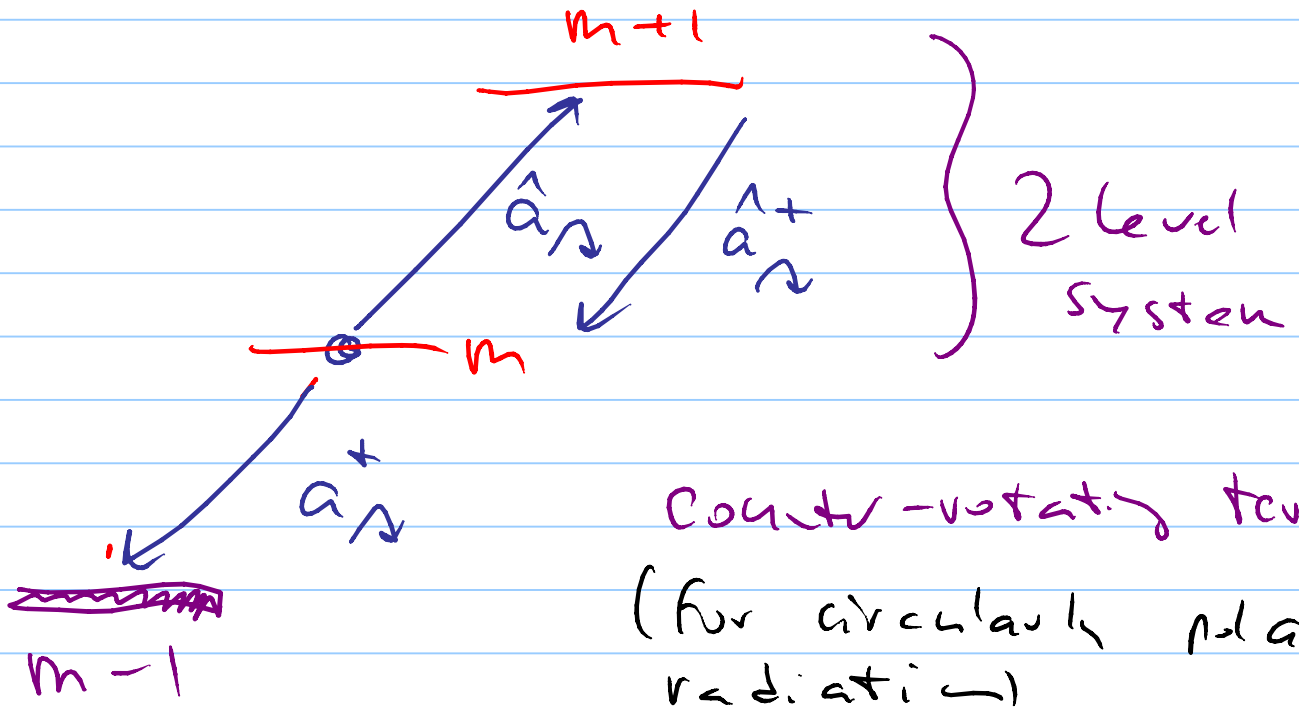
ω positive/negative for abs./em.

$\Rightarrow$ energy conservation ☺

RWA

Circular polarization

$\hat{a}_{\circlearrowright}$ $\quad$ $\hat{a}^{+}_{\circlearrowright}$



Counter-rotating term

(for circularly polarized radiation)

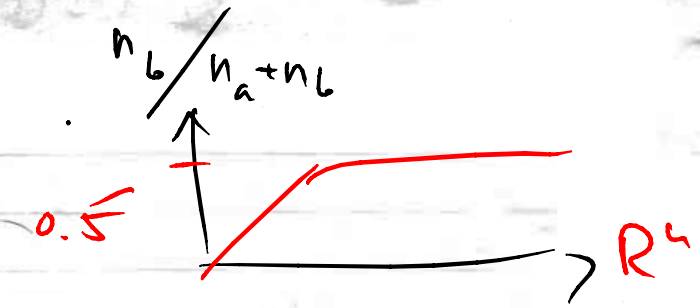
requires 3rd level —

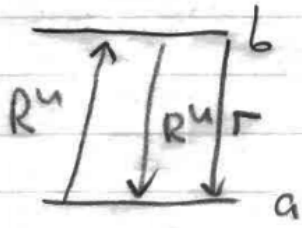
which may not exist

(but it does exist in most cases)

8.421

Saturation





R^4 Unsaturated rate of abs or stim. em

$$\dot{n}_b = -n_b(R^4+\Gamma) + R^4 n_a$$

$$\dot{n}_a = n_b(\Gamma+R^4) - R^4 n_a$$

Saturated rate R^S : net transfer $a \to b$ by e.m field per ator

$$R^S(n_a+n_b) = R^4(n_a-n_b)$$

Steady state $\frac{n_b}{n_a} = \frac{R^4}{\Gamma+R^4}$

$$\Rightarrow R^S = \frac{\Gamma}{2}\frac{S}{1+S}$$

$S \equiv 2R^4/\Gamma$ Saturation parameter

$R^S = R^4$ For $R^4 \ll \Gamma$

$R^S \to \Gamma/2$ For $S \to \infty$

Monochromatic radiation

$$R^{\downarrow} = W_{ab} = \frac{\pi}{2} w_R^2 \underbrace{f(\omega)}_{\text{normalized line shape}}$$

Fermi's Gold rule

Lorentzian

$$\Rightarrow R^{\downarrow} = \frac{w_R^2}{\Gamma} \frac{1}{1+(2\delta/\Gamma)^2}$$

$$S_{res} = 2w_R^2/\Gamma^2$$

4th eq. For R^S and S

$$R^S = \frac{w_R^2}{\Gamma} \frac{1}{1+(2\delta/\Gamma)^2 + 2w_R^2/\Gamma^2}$$

important.

Can be derived also From Optical Bloch equations

Lorentzian with

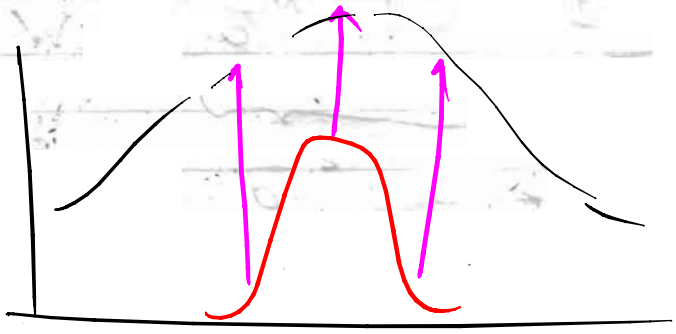
$$\delta_{FWHM} = \frac{\Gamma}{2}\sqrt{1+S_{res}}$$

Saturation (or power) broadening

$$S = S_{res}/\left(1+(2\delta/\Gamma)^2\right)$$

$$R^S = \frac{\Gamma}{2}\frac{S}{S+1}$$

Sat. means: divide rates by $1+S$



Saturation Intensity

Def: at $I = I_{SAT}$, the unsaturated rate equals $\Gamma/2$

$$S_{res} = 1 \quad \text{for} \quad R^{unsat} = \Gamma/2 = \frac{w_R^2}{\Gamma}\,\pi_{SAT}$$

$$S_{res} = \frac{I}{I_{sat}}$$

$\approx 6\,mW/cm^2$ For Na D line

$$\Rightarrow w_R^2 = \frac{\Gamma^2}{2}\frac{I}{I_{sat}}, \qquad I_{sat} = \frac{\Gamma^2}{2}\frac{I}{w_R^2} \overset{\circledast}{=} \frac{\hbar\nu^3}{12\pi c^2}\Gamma$$

or $I, \vec{d}^2$

harder to saturate at high ν (e.g. UV)

$\circledast$ express I, w_R by photon #: n

$$I = \frac{\text{energy}}{\text{area time}} = \frac{\hbar\nu\, n}{V/c} = \frac{\hbar c \nu\, n}{V}$$

$$w_R^2 = (n+1)\,w_1^2 = n\,\underbrace{(\vec{d}\cdot\hat{e})^2}_{\frac{1}{3}d^2}\left(\frac{2}{\hbar}\right)^2\left(\frac{\hbar\nu}{2\varepsilon_0 V}\right)$$

$$= \Gamma\,\frac{6\pi c^3}{V\nu^2}\,n$$

$$\frac{I}{w_R^2} = \frac{\hbar\nu^3}{\Gamma 6\pi c^2}$$

Broadband case:

$$W_{ge} = B_{ge} \frac{\bar{I}}{c} \overset{!}{=} \frac{\Gamma}{2} \quad \text{for } I = I_{SAT}$$

$$\Rightarrow \bar{I}_{SAT} = \frac{c}{2} \frac{A}{B_{ge}} = \frac{\hbar \omega_{eg}^3}{6\pi^2 c^2}$$

$$\frac{g_g}{g_e} \frac{\hbar \omega_{eg}^3}{\pi^2 c^3}$$

$\frac{1}{3}$ for $S \to P$

independent of matrix elements !

visible: $\bar{I}_{SAT} \approx \frac{12W}{cm^2} \frac{1}{cm^{-1}}$

$1\,cm^{-1} = 30\,GHz$

Cross Section

Collision rate of particles, $n \sigma v$

(single particle moving with velocity v relative to other particles with density n)

Monochromatic case

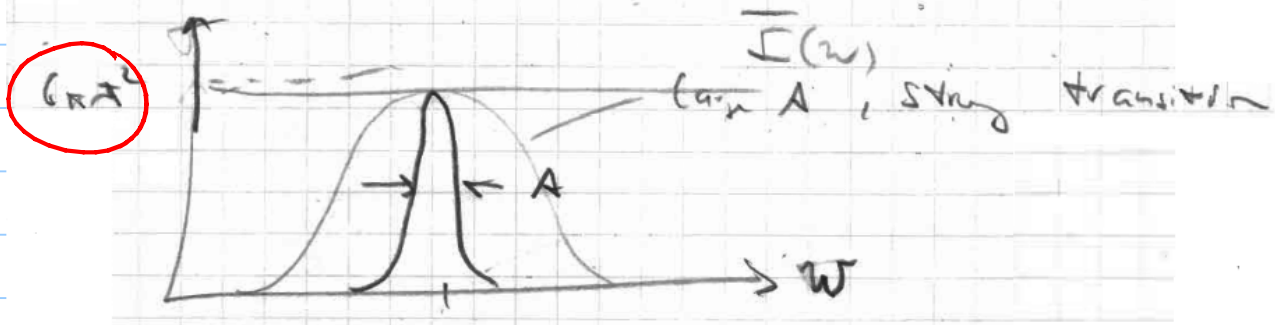
$$W_{ge} = n_{phot}\, \sigma\, c = \frac{I}{\hbar\omega \cancel{c}}\, \sigma \cancel{c}$$ | evaluated at $S=1$

$$\frac{\Gamma}{2} = \frac{I_{sat}}{\hbar\omega}\, \sigma \qquad I_{sat} = \frac{\hbar\omega^3}{12\pi c^2}\, \Gamma$$

$$\Rightarrow \sigma = \frac{c^2}{\omega^2} \cdot 6\pi = 6\pi\, \lambda\!\!\!^{-2}$$ resonant cross section

$\gg$ (size of atom)^2

independent of matrix element



strong transitions are wider, but don't have larger σ

$$W_{ge} = \underbrace{\sigma_{abs}^{u} \frac{1}{1+S}}_{\sigma_{abs}^{s} \xrightarrow[S\to\infty]{} 0} \frac{I}{\hbar\omega}$$ transition bleaches out

Broadband

$$W_{ge} = \int \sigma(\omega) \frac{\bar{I}(\omega)}{\hbar\omega} d\omega$$

$$= \frac{6\pi \lambda\!\!\!^{-2}}{\hbar \omega_{eg}} \bar{I}(\omega_{eg}) \frac{\pi \Gamma}{2}$$

integral over Lorentzian with value of 1 at max

at saturation

$$\cancel{\frac{\Gamma}{2}} = \frac{6\pi \lambda\!\!\!^{-2}}{\hbar \omega_{eg}} \bar{I}_{sat} \cancel{\frac{\pi}{2}\Gamma}$$

Saturation intensity independent of linewidth or matrix element

$$\Rightarrow \bar{I}_{sat} = \frac{\hbar \omega_{eg}}{6\pi^2 \lambda\!\!\!^{-2}} = \frac{\hbar \omega_{eg}^3}{6\pi^2 c^2}$$

as derived before

Faster rate due to better "overlap" of spectrally broad radiation and spectrally broader absorption cross section

photon # per mode

Ex: Black body radiation

Saturates ($S=1$) for $\bar{n} = \frac{1}{2}$

spont. em $\propto$ "1 photon per mode"

Abs $\propto \bar{n}$

$\bar{n} = \frac{1}{2} \Rightarrow R_4 = \Gamma/2$