PSet 4

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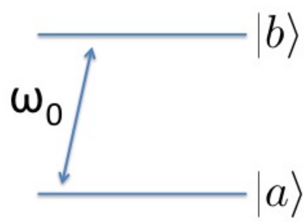
due April 27, 2016

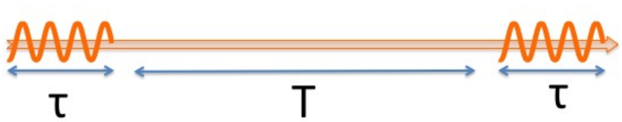
1 Ramsey Spectroscopy

Consider a two-level atom ($|a\rangle, |b\rangle$) with transition frequency ω_0. To measure this transition frequency, one can apply an EM field of frequency ω, which is varied around the atomic resonance. The initial state of the atom is $|a\rangle$.



1) What is the probability P_A that the atom makes a transition to state $|b\rangle$ if the interaction time is τ and the Rabi frequency is Ω? Write the detuning as $\delta = \omega - \omega_0$ and the generalized Rabi frequency as Ω_g.

2) Another way to probe the transition frequency is through a Ramsey sequence. First, the atom interacts with the EM field for time τ. Then, there is a period of free evolution of length T. Finally, the atom interacts with the EM field again for time τ.



We want to find the probability that an atom starting in $|a\rangle$ transitions to state $|b\rangle$ at the end of the Ramsey sequence. We start with the Hamiltonian in the rotating frame in the $\begin{bmatrix} a \\ b \end{bmatrix}$ basis:

$$H = \frac{\hbar}{2} \begin{bmatrix} -\delta & \Omega \\ \Omega & \delta \end{bmatrix} \tag{1}$$

i) Show that the solution to the Schrodinger equation for $|\Psi(t)\rangle = a(t)|a\rangle + b(t)|b\rangle$, given that

$a(t_0) = a_0$ and $b(t_0) = b_0$, is:

$$a(t+t_0) = a_0\left(\frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) + \cos\left(\frac{\Omega_g t}{2}\right)\right) - b_0\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) \tag{2}$$

$$b(t+t_0) = -a_0\frac{i\Omega}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) + b_0\left(-\frac{i\delta}{\Omega_g}\sin\left(\frac{\Omega_g t}{2}\right) + \cos\left(\frac{\Omega_g t}{2}\right)\right) \tag{3}$$

In the Ramsey method, the free evolution time is typically much longer than the interaction times. During the free evolution, there could be decoherence processes, which randomize the phase relationship between $|a\rangle$ and $|b\rangle$ but do not change the populations $|a|^2$ and $|b|^2$. This results in a statistical mixture of $|a\rangle$ and $|b\rangle$ which cannot be described by a wavefunction. In order to describe such a system, we need to use the density matrix formalism, introduced is the online PSet M3.3.

ii) Express your solution for $|\Psi(t+t_0)\rangle = a(t+t_0)|a\rangle + b(t+t_0)|b\rangle$ (eq. (2) and (3)) as a 2×2 density matrix of the form below. To simplify calculations, assume that $a_0 = 1$ and $b_0 = 0$.

$$\rho(t+t_0) = \begin{bmatrix} \rho_{aa}(t+t_0) & \rho_{ab}(t+t_0) \\ \rho_{ba}(t+t_0) & \rho_{bb}(t+t_0) \end{bmatrix} \tag{4}$$

iii) Show that you get the same $\rho(t+t_0)$ by using the Hamiltonian time evolution of the density matrix, starting from the same initial state $|\Psi(t_0)\rangle = |a\rangle$.
Hint: *The time evolution of the density matrix is*

$$i\hbar\dot{\rho} = [H, \rho] \tag{5}$$

which can also be described in terms of a transformation matrix $U = e^{-iHt/\hbar}$*, such that*

$$\rho(t) = U(t)\rho_0 U(t)^\dagger \tag{6}$$

Use $e^{i\alpha(\hat{n}\cdot\vec{\sigma})} = \mathbb{I}\cos(\alpha) + i\hat{n}\cdot\vec{\sigma}\sin(\alpha)$*, where* σ *is a Pauli matrix.*
Use Mathematica or another software to make calculations easier.

iv) Show that the transition probability to state $|b\rangle$ from $|\Psi_0\rangle = |a\rangle$ after the Ramsey sequence is:

$$P_B = \frac{4\Omega^2}{\Omega_g^2}\sin^2\left(\frac{\Omega_g\tau}{2}\right)\left[\cos\left(\frac{\Omega_g\tau}{2}\right)\cos\left(\frac{\delta T}{2}\right) - \frac{\delta}{\Omega_g}\sin\left(\frac{\Omega_g\tau}{2}\right)\sin\left(\frac{\delta T}{2}\right)\right]^2 \tag{7}$$

To do this, express the time evolution of the density matrix after the three zones in the Ramsey method in terms of products of 2×2 matrices. Use your result from above for the evolution during the interaction pulses at times $0 \to \tau$ and $(\tau+T) \to (\tau+T)+\tau$. During the free evolution time $\tau \to \tau + T$, there is no EM field and $\Omega = 0$. Note that we are only interested in $\rho_{bb}(2\tau + T)$.
Hint: *Again, use Mathematica or another software to make calculations easier. You do not lose too many points for not getting the correct result out of the matrix multiplication but be sure to write clearly which matrices you are multiplying.*

3) What should the interaction times τ be in each case so that the transition probabilities P_A (Rabi case) and P_B (Ramsey case) are maximized at $\delta = 0$? Consider only $0 \le \Omega_g\tau \le \pi$. Plot $P_A(\delta)$ and $P_B(\delta)$ for $\Omega = 2\pi \times 1\ kHz$ and $T = 1\ ms$ for the τ you have picked in each case. To

make the linewidth narrower, what can be done in the Rabi and in the Ramsey sequences? If a narrow linewidth is desired, but the EM field has inhomogeneities, which method is preferable?

4) Show that in the limit $\delta \ll \Omega$ and for the $\Omega_g \tau$ you picked in part 3), P_B simplifies to:

$$P_0 = \cos^2\left(\frac{\delta T}{2}\right) \tag{8}$$

2 Ramsey Spectroscopy with Decoherence

For this question, use $\Omega_g \tau = \pi/2$, and $\delta \ll \Omega$, so that during the interaction pulses the Hamiltonian is $H = \frac{\hbar\Omega}{2}\sigma_x$.

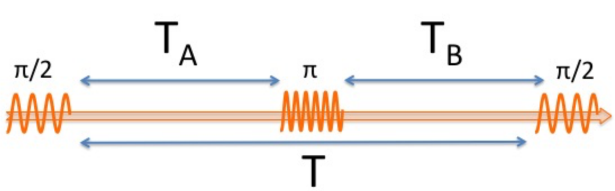
1) Homogeneous damping. Suppose that during the free evolution the coherences ρ_{ab} and ρ_{ba} decay as $e^{-\Gamma t}$. You can take this into account by modifying your result for the density matrix after the free evolution by replacing $\rho_{ab} \to \rho_{ab} e^{-\Gamma t}$ and $\rho_{ba} \to \rho_{ba} e^{-\Gamma t}$. What is the transition probability P_1 after the Ramsey sequence? Note that the decay during the short interaction pulses is negligible.

2) In addition to the homogeneous decay, there could be inhomogeneous decoherence mechanisms. These include an inhomogeneous magnetic field experienced by the localized atoms, or different Doppler shifts of atoms moving with different velocities. Model this as a spread in the distribution of detunings seen by the atom:

$$P(\delta) = \frac{1}{\sqrt{2\pi\Gamma_2^2}} e^{-\frac{(\delta-\delta_0)^2}{2\Gamma_2^2}} \tag{9}$$

What is the resulting transition probability P_2, including both homogeneous and inhomogeneous damping?

3) Consider the echo sequence shown below. The purpose of the π pulse is to time-reverse the evolution of the atoms during the free evolution. What is the transition probability P_3 in this case? Take into account both homogeneous damping of $e^{-\Gamma T}$ during the two free evolution stages of durations T_A and T_B and inhomogeneous damping, similar to part 2). Can an echo sequence cancel the homogeneous and/or the inhomogeneous decay?



Hint: *An echo fringe pattern appears for a specific choice of T_A and T_B. Plot $P_2(T)$ and $P_3(T)$ for a fixed detuning $\delta = 2\pi \times\ 200Hz$ and for $\Gamma = 5\ s^{-1}, \Gamma_2 = 40\ s^{-1}$. For $P_3(T)$, fix T_A at a few values (e.g. 50ms, 100ms, 150ms, 200ms, etc) and make $T_B = T - T_A$, so that the total evolution time T is kept the same. Can you still see fringes in the echo sequence signal P_3 for times T when the non-echo signal P_2 has decayed? Is this still the case when $\Gamma \gg \Gamma_2$?*