

Midterm Quiz

1. Zeeman shifts and hyperfine structure (24 points)

We want to look at the angular momentum structure of one electronic level of an atom.

- a. (3 p.) The atom has orbital angular momentum L , electron spin angular momentum S and nuclear angular momentum I . For very large magnetic fields B , what are the quantum numbers and energies?
- b. (3 p.) In addition to the static B field along the z axis, we add a transverse oscillating magnetic field. Which states are now coupled by the oscillating field, i.e. between which states are transitions induced?
- c. (3 p.) Assume now that $L=0$. The hyperfine interaction is $A \vec{S} \cdot \vec{I}$. What are the energy levels (to first order in A)?
- d. (3 p.) For $S=1/2$, $I=1/2$ plot the energy levels vs. magnetic fields. Assume $A>0$. Label all levels with their quantum numbers. Show the transitions induced by the transverse field.
- e. (3 p) For each allowed transition, what is the relevant matrix element (ignore numerical factors of 2 or similar)?
- f. (6 p) Back to the situation in c ($L=0$, arbitrary S and I): What are the eigenstates in first order of the hyperfine Hamiltonian? Express the eigenstates by the unperturbed states (i.e. states at infinite magnetic field), the angular momentum quantum numbers, A and B .
- g. (3 p) Using the first order wavefunctions: For $S=1/2$, $I=1/2$, discuss whether additional transitions are allowed now. For all possible transitions, give the relevant matrix elements (again, neglecting numerical factors).

2. Cavity QED (16 points)

- a. (2 p) An atom is coupled to a single mode of a cavity using the quantized electromagnetic field. For negligible coupling between the atoms and the electromagnetic field, what is the ground state of the total system?
- b. (6 p) For finite coupling, what is the ground state and its energy in leading order of the coupling?
- c. (2 p) For atoms in free space, is there an analogous effect?
- d. (6 p) At time $t=0$, the atom is in the excited state, and the cavity mode (at frequency ω different from ω_{eg}) is in a superposition state of photon number states $|n\rangle$: $a_1 |1\rangle + a_2 |2\rangle$. What is the state of the total system as a function of time? (Assume that the detuning $\omega - \omega_{eg}$ is sufficiently large so that you can neglect depletion of the initial atomic state. Use the RWA).

SHORT QUESTIONS

3. AC Stark effect and spontaneous emission (4 points)

a. (2 p) An alternating electric field at frequency ω very different from the resonant frequency ω_{eg} admixes 10 % population of the excited state to the ground state. The excited state $|e\rangle$ has a lifetime τ . What is the rate of spontaneous emission of the lowest energy state in the presence of the oscillating electric field? Assume validity of the rotating wave approximation.

b. (2 p) For very large drive fields, what is the wavefunction of the lowest state? What is the scattering rate?

4. Hyperfine structure for an exotic atom (6 points)

Physicist have found a new positively charged particle with spin $\frac{1}{2}$ and a magnetic moment of 2 bohr magneton (i.e. twice the value of the positron). This particle, together with a normal electron, forms a hydrogen-like atom. Discuss for low and high magnetic fields: what are the magnetic moments of the hyperfine states of the 1s ground state?

5. Relativistic corrections (4 points)

What is the characteristic velocity of an electron in the ground state of hydrogen (ignore numerical factors)?

What is the velocity in a Rydberg state with large principal quantum number n for a highly charged nucleus (charge Z)?

If you replace the electron by a negative muon (which is approximately 200 times heavier), how does this affect the characteristic velocity?

Formulae sheet

- Angular momentum operators

$$\underline{J}_+ = J_x + i J_y$$

$$\underline{J}_+ |j m\rangle = \sqrt{(j-m)(j+m+1)} |j m+1\rangle$$

- Perturbation theory

$$|n\rangle = |n^{(0)}\rangle + \sum_{k \neq n} \frac{\langle k^{(0)} | V | n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

$$E_n^{(1)} = E_n^{(0)} + \langle n^{(0)} | V | n^{(0)} \rangle + \sum_{k \neq n} \frac{|\langle k^{(0)} | V | n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

↑
2nd order

- Atom coupled to single-mode cavity @ ω

$$\mathcal{H} = \frac{1}{2} \hbar \omega_c \hat{\sigma}_z - \frac{1}{2} \hbar \omega_r (\hat{\sigma}_+ + \hat{\sigma}_-) (\hat{a} + \hat{a}^\dagger) + \hbar \omega \hat{a}^\dagger \hat{a}$$

$$\hat{\sigma}_+ = |e\rangle \langle g| \quad \hat{\sigma}_- = |g\rangle \langle e|$$

- Rabi oscillation

$$a_g |g\rangle + a_e |e\rangle$$

$$a_e = \frac{\omega_R}{\Omega_R} \sin\left(\frac{1}{2} \Omega_R t\right)$$

$$\Delta = \omega - \omega_c$$

$$\Omega_R = \sqrt{\omega_R^2 + \Delta^2}$$

- Rydberg energy: $\frac{1}{2} \frac{m e^4}{\hbar^2}$
for hydrogen

Quiz Solutions

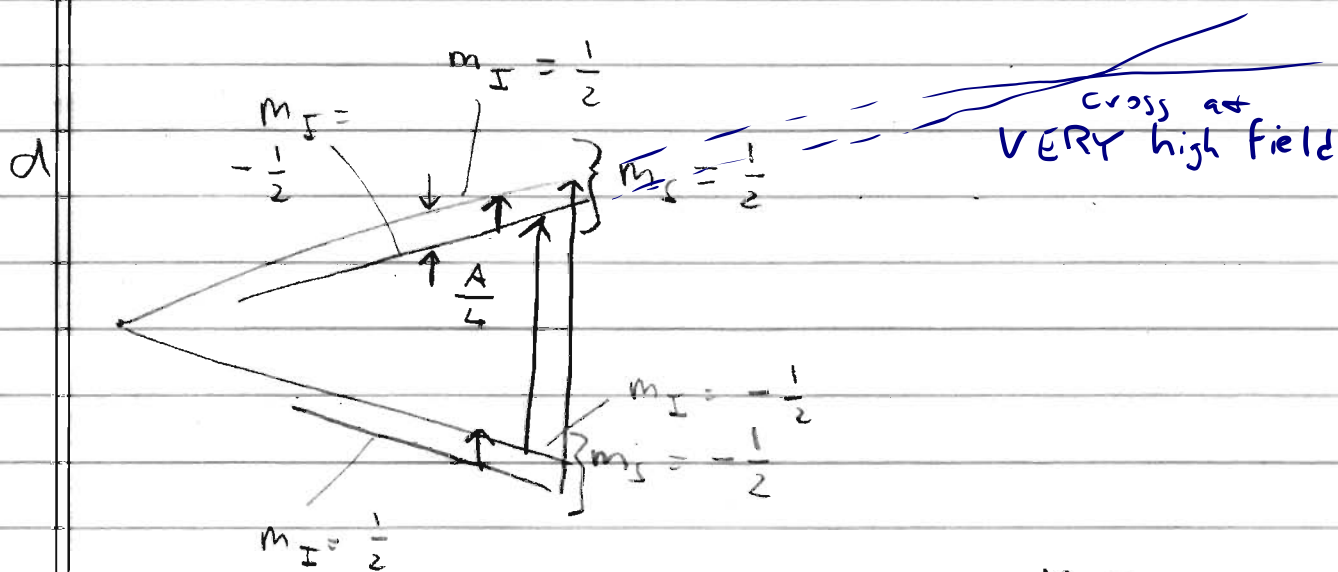
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1 a) quantum numbers m_e, m_s, m_I

$$E = B (\mu_B (m_e + g_e m_s) - \mu_N g_N m_I)$$

b) $\Delta m_e = \pm 1$ or $\Delta m_s = \pm 1$ or $\Delta m_I = \pm 1$
only one quantum number changes

c) $E = B (\mu_B g_e m_s - \mu_N g_N m_I) + A m_s m_I$



e) Four transitions

$$\Delta m_s = \pm 1$$

Matrix element

$$g_e \mu_B$$

$$\Delta m_I = \pm 1$$

$$g_N \mu_N$$

4) HF term:

$$A \vec{S} \cdot \vec{I} = A \left(\underbrace{S_z I_z}_{\text{diagonal}} + \frac{1}{2} S_+ I_- + \frac{1}{2} S_- I_+ \right)$$

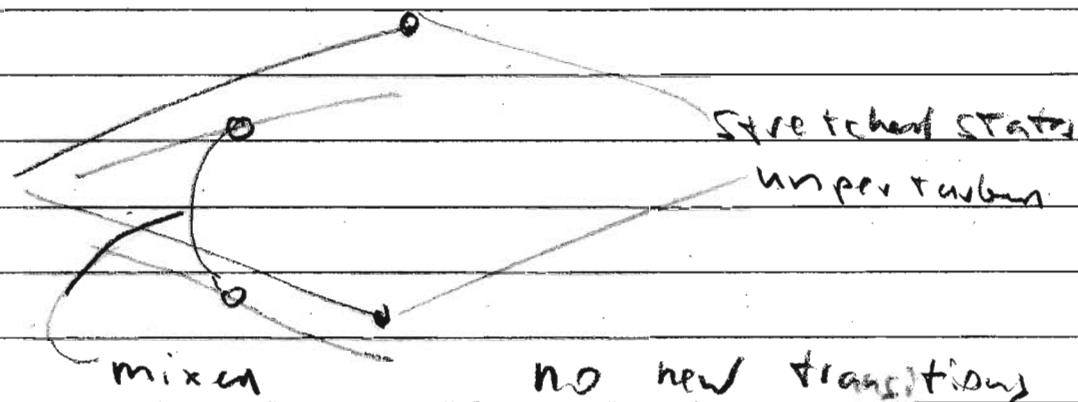
perturbed eigenstates

$$|m_s, m_I\rangle = |m_s, m_I\rangle + \frac{A}{2} \frac{[(S-m_s)(S+m_s+1)(I+m_I)(I-m_I+1)]^{1/2}}{B\mu_B g_e + \mu_N g_N B} |m_s+1, m_I-1\rangle$$

$$+ \frac{A}{2} \frac{[(S+m_s)(S-m_s+1)(I-m_I)(I+m_I+1)]^{1/2}}{-B\mu_B g_e - \mu_N g_N B} |m_s-1, m_I+1\rangle$$

5) HF interaction mixed

$$m_s = +\frac{1}{2}, m_I = -\frac{1}{2} \quad \text{and} \quad m_s = -\frac{1}{2}, m_I = +\frac{1}{2}$$



no new transitions possible with

$$\Delta m_s = \pm 1 \quad \text{or} \quad \Delta m_I = \pm 1$$

$\Delta m_s = \pm 1$ transitions: matrix element $g_e \mu_B$

$\Delta m_I = \pm 1$ transitions: $g_N \mu_N + g_e \mu_B \frac{A}{B\mu_B g_e}$

2.) Cavity QED

a) $|G\rangle = |g\rangle \otimes |0\rangle$

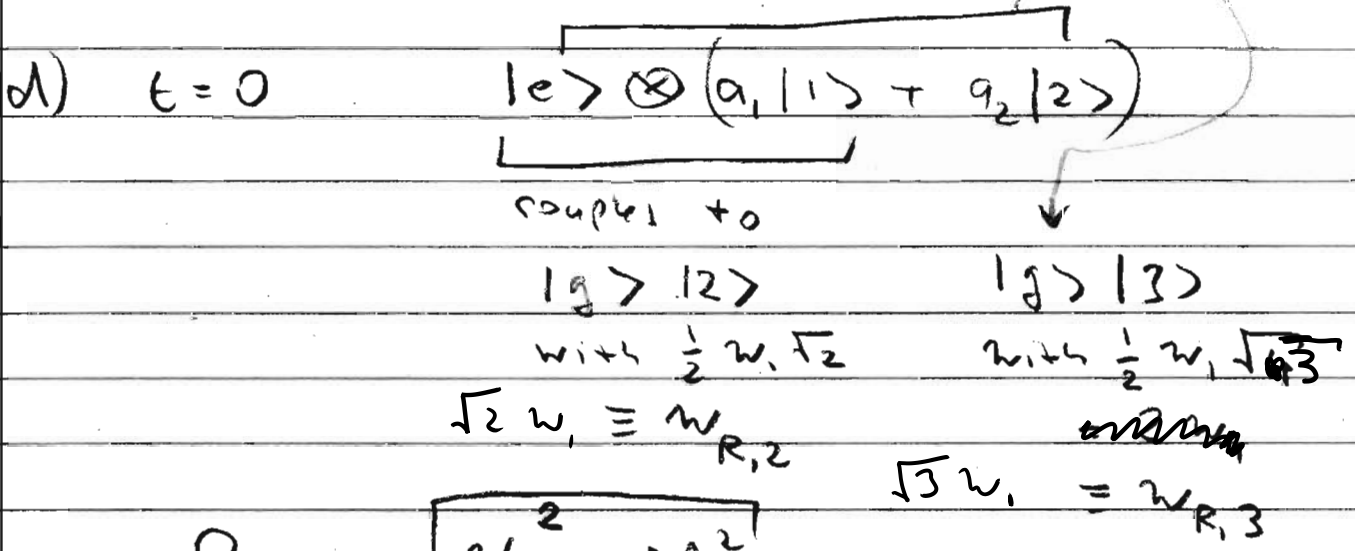
b) $|G\rangle$ is coupled to $|e\rangle \otimes |1\rangle$
 by $-\frac{1}{2} \hbar \omega_1 \hat{\sigma}_+ \hat{a}^\dagger = V$

$(\langle e| \otimes \langle 1| | V | |g\rangle \otimes |0\rangle) = -\frac{1}{2} \hbar \omega_1$

$|G\rangle' = |G\rangle + \frac{\frac{1}{2} \hbar \omega_1}{\omega_{eg} + \omega} |e\rangle \otimes |1\rangle$

$E_G = \frac{\frac{1}{4} \omega_1^2 \hbar}{\omega_{eg} + \omega}$

c) Yes, the Lamb shift



Wave Function $|e\rangle \otimes (a_1 |1\rangle + a_2 |2\rangle) +$
 $a_1 \frac{\omega_{R,2}}{\Omega_{R,2}} \sin(\frac{1}{2} \Omega_{R,2} t) |g\rangle \otimes |2\rangle +$

$$+ a_2 \frac{\omega_{R,3}}{\Omega_{R,3}} \sin\left(\frac{1}{2} \Omega_{R,3} t\right) |g\rangle \otimes |3\rangle$$

3) a) Rate = $0.1 \Gamma \frac{\omega^3}{\omega_{eg}^3}$

b) $|2\rangle = \frac{1}{\sqrt{2}} (|g\rangle + e^{i\omega t} |e\rangle)$

Rate = $\frac{1}{2} \Gamma \frac{\omega^3}{\omega_{eg}^3}$

not fully correct
since we have to use
a dressed atom
description

4) high Field

m_n	m_e	magnetic moment [μ_B]	
$\frac{1}{2}$	$\frac{1}{2}$	2 - 1	1
$\frac{1}{2}$	$-\frac{1}{2}$	2 + 1	3
$-\frac{1}{2}$	$\frac{1}{2}$	-2 - 1	-3
$-\frac{1}{2}$	$-\frac{1}{2}$	-2 + 1	-1

low Field:

two stretched states with $M = m_n + m_e = \pm 1$

magnetic moment $\pm 1 \mu_B$
 $\hat{=} |F, m_F\rangle = |1, \pm 1\rangle$

two states $|1, 0\rangle, |0, 0\rangle$

with zero magnetic moment

$$5) \quad v = \alpha c \quad \text{hydrog}$$

$$v = \alpha c \frac{Z}{n}$$

Kinetic energy $\propto \frac{Z^2}{n^2}$ Rydberg Formula

$$\frac{1}{2} \frac{m}{M} \frac{v^2}{M} \propto \frac{Ryd}{M} \propto \frac{m}{M}$$

Velocity independent of mass