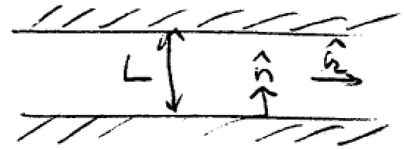
L 11

# CASIMIR EFFECT

(cf. S. Haroche Les Houches Summer School)

Attractive force between two parallel conducting plates due to vacuum fluctuations of the e.m. field

e.m modes in cavity width $L$, cross section $a^2$, $a \gg L$




boundary conditions at $z=0$ and $z=L$

$$\sin L\ell = 0$$
$$\ell = m\pi/L \qquad m = 0, 1, 2, \dots$$
$$\omega^2 = k^2c^2 + m^2\pi^2c^2/L^2$$

TE modes: $$\vec{A}^{E}_{\vec{k},m} = \sin\left(\frac{m\pi z}{L}\right)\left(\hat{k}\times\hat{n}\right)e^{ik\hat{k}\vec{s}} + c.c. \qquad m \geq 1$$

TM modes $$\vec{A}^{M}_{\vec{k},m} = \left\{\left[\frac{ck}{\omega}\cos\left(\frac{m\pi z}{L}\right)\hat{n} - \frac{im\pi c}{L\omega}\sin\left(\frac{m\pi z}{L}\right)\hat{k}\right.\right.$$
$$\left.\times\, e^{ik\hat{k}\vec{s}}\right\} + c.c. \qquad m \geq 0$$

→ Density of states

– number of modes with given $m$ and $k$ between $k$ and $k + dk$

$$\frac{dk_x}{\left(\frac{2\pi}{a}\right)}\frac{dk_y}{\left(\frac{2\pi}{a}\right)} = \frac{a^2\, k\, dk}{2\pi} = \frac{a^2}{4\pi}\,d(k^2) = \frac{a^2}{c^2}\frac{\omega\, d\omega}{2\pi}$$

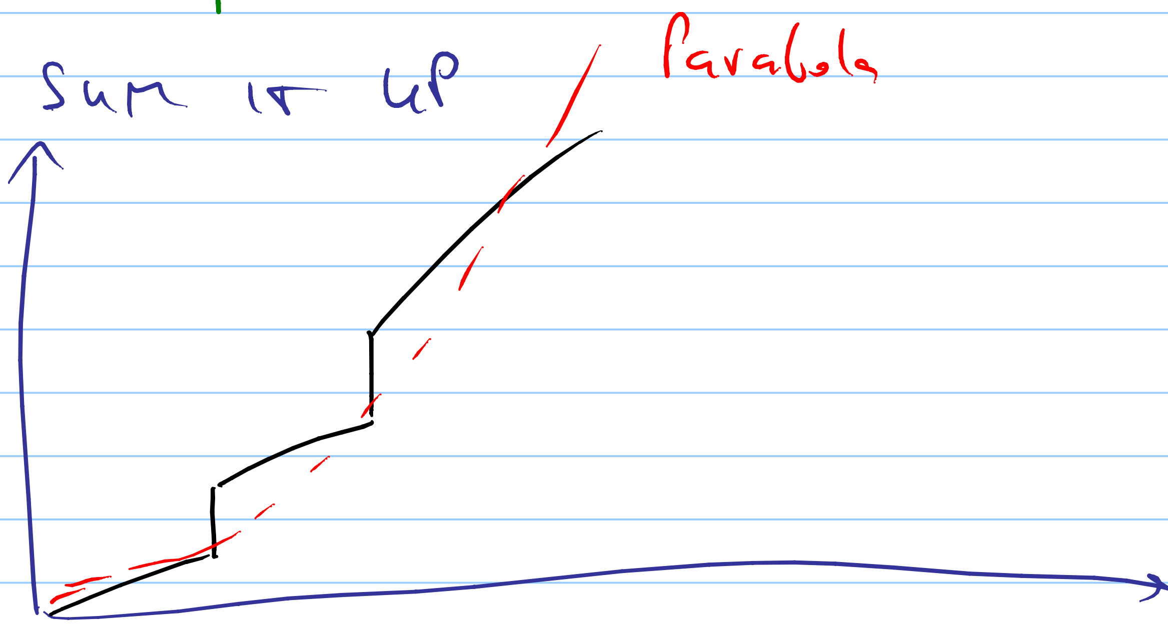
– for a given $\omega$: $m = 0, \dots, \mathrm{Int}\left(\frac{\omega L}{\pi c}\right)$

$$\rho(\omega)\,d\omega = \frac{a^2}{2\pi c^2}\left[1 + 2\,\mathrm{Int}\left(\frac{\omega L}{\pi c}\right)\right]\omega\, d\omega$$
$$= \frac{a^2}{2\pi c^2}\left[1 + 2\sum_{m=1}^{\infty}\Theta\left(\omega - \frac{m\pi c}{L}\right)\right]\omega\, d\omega$$

Heaviside step function

Density of modes for a given m $(\;)\,\omega\, d\omega$


Sum it up



Zero-point energy in the volume $a^2 \times L$

$$W(L) = \int_0^\infty \frac{\hbar w}{2} \rho(w)\, dw$$

$$= \frac{a^2 \hbar}{4\pi c^2}\left[\int_0^\infty w^2 dw + 2\sum_{m=1}^{\infty}\int_{\frac{m\pi c}{L}}^{\infty} w^2 dw\right]$$

add convergence term $e^{-\lambda w/c}$
[cutoff at high frequencies $w \gg \lambda/c$]

$$W(L) = \text{const.} + \frac{a^2 \hbar}{2\pi c^2}\sum_m I_m$$

with $I_m = \int_{\frac{m\pi c}{L}}^{\infty} w^2 e^{-\lambda w/c} dw$

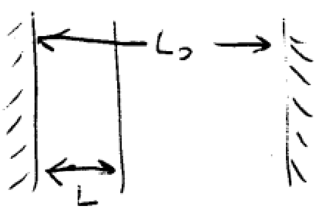
$$I_m = c^2 \frac{\partial^2}{\partial\lambda^2}\int_{\frac{m\pi c}{L}}^{\infty} e^{-\lambda w/c} dw = c^3 \frac{\partial^2}{\partial\lambda^2}\left[-\frac{e^{-m\pi\lambda/L}}{\lambda}\right]$$

$$\sum_m I_m = \frac{c^3 \pi}{L}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{\left(\frac{\pi\lambda}{L}\right)}\frac{1}{e^{(\pi\lambda/L)}-1}\right]$$

Expansion around $x=0$ $\quad$ ($x = \pi\lambda/L$, eventually $\lambda \to 0$)

$$\frac{1}{x(e^x-1)} = \frac{1}{x^2} - \frac{1}{2x} + \frac{1}{12} - \frac{x^2}{720} + \ldots$$

$$W(L) = W_0 + \frac{a^2\hbar c}{2}\left[\frac{6L}{\pi^2\lambda^4} - \frac{1}{\pi\lambda^3} - \frac{2\pi^2}{720 L^3} + \ldots\right]$$




$$W_T = W(L) + W(L_0 - L) = \underbrace{\qquad}_{} \text{indep. of } L$$

$$= \frac{a^2\hbar c}{2}\left[\overbrace{\frac{6L}{\pi^2\lambda^4} + \frac{6(L_0-L)}{\pi^2\lambda^4}} - \frac{2}{\pi\lambda^3} - \frac{2\pi^2}{720}\left(\frac{1}{L^3} + \frac{1}{(L_0-L)^3}\right) + \ldots\right]$$

$10^{-5}$ mbar @ $L = 1\,\mu m$

$\lambda$ independent potential energy

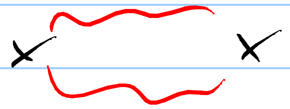
$$U(L) = -\frac{\pi^2\hbar c}{720} a^2 \frac{1}{L^3}$$

[Casimir energy]

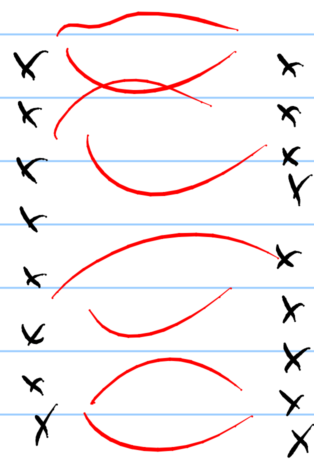
$$P_{vacuum} = \frac{1}{a^2}\frac{\partial U}{\partial L} = \frac{\pi^2\hbar c}{240}\frac{1}{L^4}$$

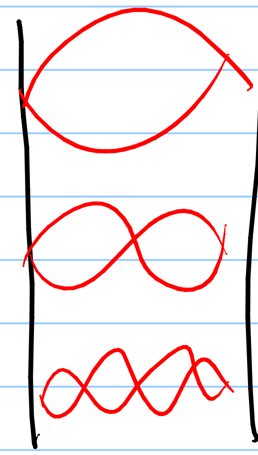
# Discussion

Two ways to derive the Casimir potential



two atoms





$\sum V_{\text{atom-atom}}$ $\qquad$ $\sum \frac{\hbar\omega}{2}$

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Lamoreaux, Phys. Tod

both results are equivalent

is relevant for dark energy

$$\rho_E = \sum \frac{\hbar\omega}{2} = \frac{\hbar c}{2\pi}\int k^3\, dk = \frac{\hbar c}{8\pi} k_{max}^4$$

energy density

$$k_{max} = \frac{2\pi}{r_e}$$ Classical $e^-$ radius $10^{43}$

$$k_{max} = \frac{2\pi}{l_p}$$ $l_p = \sqrt{\hbar G / c^3}$ $10^{124}$

× dark energy

Jaffe:

Casimir force is independent of atomic properties, in the limit $\alpha \to \infty$

[$\alpha \to 0$, $a_0 \to \infty$, Casimir force $\to 0$]

For $d \leq 0.5\,\mu m$

$\alpha \gtrsim 10^{-5}$ $(\alpha = 1/137)$

Refs: Wiki.

Jaffe paper

Haroche "

Lamoreaux " (2007)