1. The Hanbury-Brown Twiss experiment and $g^{(2)}(\tau)$

The second-order coherence function $g^{(2)}(\tau)$ is often measured in the laboratory using an experiment first developed by Hanbury-Brown and Twiss in the 1950’s, for studying the light from distant stars. This experiment involves mixing light from the input source with the vacuum, $|0\rangle$, on a 50/50 beamsplitter, and measuring the intensity-intensity correlation function at the output using two detectors and a coincidence circuit:

This problem examines how this experiment measures $g^{(2)}(\tau)$, and what results are obtained for different input states of light.

a) Let $a$, $a^\dagger$, and $b$, $b^\dagger$ be the raising and lowering operators for the two modes of light input to the beamsplitter, and let the unitary transformation performed by the beamsplitter be defined by

$$ a_1 = Ua U^\dagger = \frac{a + b}{\sqrt{2}} \quad (1) $$

$$ b_1 = Ub U^\dagger = \frac{a - b}{\sqrt{2}}. \quad (2) $$

For light input in state $|\psi\rangle$, you are given that the output of the coincidence circuit is a voltage

$$ V_{\psi} = V_0 \langle \psi, 0 | a_1^\dagger a_1 b_1^\dagger b_1 | \psi, 0 \rangle, \quad (3) $$

where $V_0$ is some proportionality constant, and $|\psi, 0\rangle$ denotes a state with $|\psi\rangle$ in mode “$a$” and $|0\rangle$ in mode “$b$”. In other words, the voltage is the average of the product of the two detected photon signals. Show that $V_{\psi}$ gives a measure of $g^{(2)}(\tau)$,

$$ g^{(2)}(\tau) = \frac{\langle a_1^\dagger a_1 b_1^\dagger b_1 \rangle}{\langle a_1^\dagger a_1 \rangle^2} \quad (4) $$

up to a normalization factor. Real detectors will have an additional additive offset (due to dark currents etc.) which we ignore here.
b) The classical expression for $g^{(2)}$ is
\[ g^{(2)}_{cl}(\tau) = \frac{\langle \bar{I}(t) \bar{I}(t+\tau) \rangle}{\langle I \rangle^2}. \] (5)

Prove that $g^{(2)}_{cl}(0) \geq 1$. It is helpful to use the fact that $\langle (\bar{I}(t) - \langle \bar{I}(t) \rangle)^2 \rangle \geq 0$.

c) Compute $g^{(2)}(\tau)$ for the following input light states:
   \[ |\psi_1\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_k \frac{\alpha^k}{\sqrt{k!}} |k\rangle \quad \text{(a coherent state)} \] (6)

   \[ |\psi_2\rangle = |2\rangle \quad \text{(the number state } n = 2) \] (7)

d) Compute $g^{(2)}(\tau)$ for the following input light states, as a function of $\alpha$:
   \[ |\psi_3\rangle = \frac{|\alpha\rangle + | - \alpha\rangle}{\sqrt{2}} \] (8)

   \[ |\psi_4\rangle = \frac{|\alpha\rangle - | - \alpha\rangle}{\sqrt{2}} \] (9)

Do either of these two states show non-classical second-order coherence? Why (or why not)?

2. Hanbury Brown and Twiss Experiment with Atoms

This problem illustrates the coherence and collimation requirements for performing a Hanbury Brown and Twiss (HBT) experiment with atoms. In fact the HBT experiment was done for both bosons ($^4$He) and fermions ($^3$He) by Jeltes and company in 2007 (T. Jeltes et al., Nature 445, 402 (2007)). (Note: Ignore gravity in this problem.)

If a free particle starts at point $A$ at time $t = 0$ with an amplitude (wavefunction) $\psi_A$, then the amplitude at another point 1 and time $t = \tau$ is proportional to $\psi_A e^{i(k \cdot r_{A1} - \omega \tau)}$, where $r_{A1}$ is the vector from $A$ to 1, $k$ is the particle’s wavevector, and $\hbar \omega$ is its total energy. This can be regarded as Huygen’s principle for matter waves, and is a special case of the Feynman path integral formulation of quantum mechanics.

(Based on figure 19-5, in G. Baym, Lectures on Quantum Mechanics)
(a) Correlation function
Assume we have a particle at \( A \) with amplitude \( \psi_A \) and one at \( B \) with amplitude \( \psi_B \). The joint probability \( P \) of finding one particle at 1 and one at 2 is

\[
P = |\psi_A e^{i\phi_{A1}}\psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}}\psi_B e^{i\phi_{B1}}|^2
\]

and is proportional to the second-order coherence function \( g^{(2)}(1, 2) \). The \( \pm \) is for bosons/fermions and makes the two-particle wavefunction symmetric/antisymmetric under the exchange of particles. Here, \( \phi_{A1} = k_A \cdot r_{A1} - \omega \tau \) is the phase factor for the path from point \( A \) to detector 1, etc. Calculate \( P \) as a function of \( r_{21} \), the vector from point 2 to point 1 on the detector.

(b) Transverse Collimation
Assume you are given a source (e.g. a ball of trapped atoms) with transverse dimension \( W \) and detector with transverse dimension \( w \) where \( |r_{21}| \leq w \). The distance between source and detector \( d \) is much greater than all other distances. The transverse component of the phase factor in part (a) can be written: \( \phi_t = (k_A - k_B)_t \cdot (r_{21})_t \). Assume that the signal at the detector is mainly due to atoms with wavevectors distributed around \( k_0 \). Argue that the transverse collimation required to see second order correlation effects can be expressed as \( Ww \ll dB \lambda \), where \( \lambda dB \) is the deBroglie wavelength corresponding to \( k_0 \). (Hint: How does \( \phi_t \) vary for atoms originating at different points in the source and being detected at different points on the detector?) Consider a \(^6\)Li MOT at 500 \( \mu \)K. Calculate the deBroglie wavelength. Assuming a MOT and detector of approximately equal size \( (W \approx w) \), estimate an upper bound on the MOT and detector size using \( d = 10 \) cm.

(c) Longitudinal Collimation
(i) The longitudinal component of the phase factor in part (a) can be written: \( \phi_l = (k_A - k_B)_l \cdot (r_{21})_l \). Assume a Gaussian distribution of wavevector differences \( p(k_A - k_B) = e^{-|k_A - k_B|^2/\gamma^2} \) where the width \( \gamma \) is related to the temperature of the atoms. Calculate \( \langle P \rangle \) using this distribution and your result from part (a). Sketch \( \langle P \rangle \) for both fermions and bosons, indicating the extent of \( (r_{21})_l \) over which the second order correlation effect can be seen. (Hint: Use the fact that \( \phi_l \ll 2\pi \) from part (b) to simplify the integral.)

(ii) Now assume you have a pulsed source of atoms with longitudinal dimension \( L \). Atoms are released at time \( t = 0 \) and detected at some later time \( t = \tau \). Give geometric arguments to show that the wavevectors of detected atoms must obey \( |(k_A - k_B)_l| \leq \frac{\gamma\sqrt{Ld}}{v \tau} \), where the velocity \( v = \frac{d}{\tau} \). This implies that the different velocity groups separate during the expansion, narrowing (by a factor \( \frac{L}{d} \)) the velocity distribution of atoms detected at any particular time.

Consider again the \(^6\)Li MOT from part (b). Assuming \( \tau = 0.1 \)s and \( L \approx W \), estimate the necessary timing resolution of the detector in order to see second order correlation effects?

(d) Phase-Space Volume Enhancement
We now pull all the pieces together. The peak in \( g^{(2)}(1, 2) \) is visible for \( (k_A - k_B) \cdot r_{21} \leq 2\pi \). This is equivalent to saying that we must detect atoms from within a single phase
space cell, defined by $\delta p_x \delta x \leq h$ (and likewise for $y$ and $z$). In our trapped atom sample, the 3D volume of a phase space cell is $\delta x \delta y \delta z = (\lambda dB)^3$. Liouville’s theorem says that as our ball of atoms expands, the number of phase space cells remains constant. Verify that, by using this pulsed source, the volume of a coherent phase space cell is increased by a factor $d^3/W^2 L$ by the time atoms reach the detector. What is the order of magnitude of this increase (assuming $L \approx W$)?

Estimate the average occupation of a cell of phase space for the $^6$Li MOT from parts (b) and (c). Use the following numbers for the $^6$Li MOT: $10^{10}$ atoms in $1 \text{ cm}^3$. How does this compare with the average occupation of a BEC or a degenerate Fermi cloud?