# Assignment #1 Solutions

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1. **The Hanbury-Brown Twiss experiment and** $g^{(2)}(\tau)$

**a)** Using the definition of $a_1$ and $b_1$ , we find

$$\langle\psi,0 \mid {a_1}^\dagger a_1 {b_1}^\dagger b_1 \mid \psi,0\rangle \tag{1}$$

$$= \langle\psi,0 \mid \frac{a^\dagger + b^\dagger}{\sqrt{2}} \frac{a+b}{\sqrt{2}} \frac{a^\dagger - b^\dagger}{\sqrt{2}} \frac{a-b}{\sqrt{2}} \mid \psi,0\rangle \tag{2}$$

$$= \langle\psi,0 \mid \frac{a^\dagger}{\sqrt{2}} \frac{a+b}{\sqrt{2}} \frac{a^\dagger - b^\dagger}{\sqrt{2}} \frac{a}{\sqrt{2}} \mid \psi,0\rangle \tag{3}$$

$$= \frac{1}{4}\langle\psi,0 \mid (a^\dagger a a^\dagger a - a^\dagger b b^\dagger a) \mid \psi,0\rangle \tag{4}$$

$$= \frac{1}{4}\langle\psi,0 \mid (a^\dagger(a^\dagger a + 1)a - a^\dagger a) \mid \psi,0\rangle \tag{5}$$

$$= \frac{1}{4}\langle\psi,0 \mid (a^\dagger a^\dagger a a) \mid \psi,0\rangle \tag{6}$$

$$\tag{7}$$

In above calculations, we use the following properties of creation and annihilation operators:

$$\langle\psi,0 \mid b^\dagger = 0 \tag{8}$$

$$b \mid \psi,0\rangle = 0 \tag{9}$$

$$[a, a^\dagger] = 1 \tag{10}$$

$$\langle\psi,0 \mid (b b^\dagger) \mid \psi,0\rangle = 1 \tag{11}$$

$$\tag{12}$$

Therefore, the voltage $V_\psi$ gives us,

$$V_\psi = \frac{V_0}{4}\langle\psi,0 \mid (a^\dagger a^\dagger a a) \mid \psi,0\rangle \tag{13}$$

This shows that $V_\psi$ gives a measure of the second order coherence up to an additive offset, and normalization.

2. **Hanbury Brown and Twiss Experiment with Atoms**

(a) Correlation function This part is fairly straightforward substitution,

$$\begin{aligned} P &= \left|\psi_A e^{i\phi_{A1}}\psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}}\psi_B e^{i\phi_{B1}}\right|^2 \\ &= \left|\psi_A\psi_B\right|^2 \left|e^{i(\mathbf{k}_A\cdot\mathbf{r}_{A1}-\omega\tau+\mathbf{k}_B\cdot\mathbf{r}_{B2}-\omega\tau)} \pm e^{i(\mathbf{k}_A\cdot\mathbf{r}_{A2}-\omega\tau+\mathbf{k}_B\cdot\mathbf{r}_{B1}-\omega\tau)}\right|^2 \\ &= 2\left|\psi_A\psi_B\right|^2 \left[1 \pm \cos(\mathbf{k}_A\cdot(\mathbf{r}_{A2}-\mathbf{r}_{A1}) + \mathbf{k}_B\cdot(\mathbf{r}_{A1}-\mathbf{r}_{A2}))\right] \\ &= 2\left|\psi_A\psi_B\right|^2 \left[1 \pm \cos\left((\mathbf{k}_A-\mathbf{k}_B)\cdot\mathbf{r}_{21}\right)\right] \end{aligned} \tag{14}$$

(b) Transverse Collimation

What does it mean to "see a second-order correlation effect"? As discussed in class, the second order correlation function is a measure of the probability of detecting *two* particles (e.g. atoms or photons) within a certain distance/time of each other. Due to the Pauli exclusion principle, it should be relatively *less* likely to detect two fermions 'close' to each other in space or time, and due to bosonic enhancement, relatively *more* likely to detect two bosons 'close' to each other. We can see this explicitly occurring in Equation (14). As long as the distance between points of detection, $\mathbf{r}_{21}$ is small enough that $(\mathbf{k}_A - \mathbf{k}_B)\cdot\mathbf{r}_{21} \equiv \phi_t \ll 2\pi \implies \cos \simeq 1$ it indicates an increased probability of detecting two bosons and a decreased probability of detecting two fermions.

When we perform an actual experiment, unless $\phi_t \ll 2\pi$ for all the different wavevector pairs coming from the source and all the different possible detected positions, the cosine term will average to zero and we will see no difference in probability for bosons versus fermions. The maximum difference in transverse wavevector, for particles coming opposite edges of the cloud but arriving at nearly the same point on the detector, is $(\mathbf{k}_A - \mathbf{k}_B)_{max} \simeq k_0\frac{W}{d}$. The maximum separation between detected particles is given by the width of the detector $(\mathbf{r}_{21})_{max} \simeq w$. So to ensure that $\phi_t \ll 2\pi$, we must have: $k_0\frac{W}{d}w \ll 2\pi$,or

$$Ww \ll \lambda_{dB} d \text{ where } \lambda_{dB} = \frac{2\pi}{k_0}.$$

Another way to approach the problem is to determine under what conditions the two particles being detected lie within a single phase space cell, (i.e. $\delta x\delta p \ll h$) which, roughly speaking, means that they are detected in the same quantum state. Since two fermions cannot occupy the same quantum state, we would expect to see second order correlation effects in this case. Substituting $\mathbf{r}_{21}$ for $\delta x$ and $\hbar(\mathbf{k}_A - \mathbf{k}_B)$ for $\delta p$, it is clear that we will arrive at the same result given above.

The deBroglie wavelength at 500 $\mu$K is $\lambda_{dB} = h/\sqrt{2\pi m k_B T} = 32$ nm. Given $d = 10$cm, this means $W, w \ll 56\mu$m.

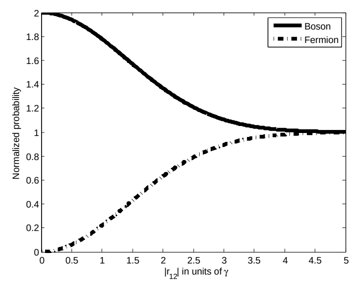
(c) Longitudinal Collimation

($i$) Average $P$ over the given wavevector distribution, normalizing appropriately:

$$\langle P\rangle = 2\left|\psi_A\psi_B\right|^2 \frac{\int_{-\infty}^{\infty}\left(1 \pm \cos\left((\mathbf{k}_A-\mathbf{k}_B)\cdot\mathbf{r}_{21}\right)\right)e^{-|\mathbf{k}_A-\mathbf{k}_B|^2\gamma^2}d(\mathbf{k}_A-\mathbf{k}_B)}{\int_{-\infty}^{\infty} e^{-|\mathbf{k}_A-\mathbf{k}_B|^2\gamma^2}d(\mathbf{k}_A-\mathbf{k}_B)}.$$

We could proceed by integrating separately over the transverse and longitudinal wavevector components: $d(\mathbf{k}_A - \mathbf{k}_B) = d(\mathbf{k}_A - \mathbf{k}_B)_t d(\mathbf{k}_A - \mathbf{k}_B)_l$, but we can avoid some work by using the condition we derived above $\phi_t \equiv (\mathbf{k}_A - \mathbf{k}_B)_t \cdot (\mathbf{r}_{21})_t \ll 2\pi$. This means we can neglect any transverse contribution to the cosine, leaving numerator and denominator with the same $\phi_t$ dependence, and therefore cancels. The remaining (longitudinal) integral is straightforward:

$$\langle P \rangle = 2\left|\psi_A \psi_B\right|^2 \left[1 \pm \frac{\int_{-\infty}^{\infty} \cos\left(\delta k_l (\mathbf{r}_{21})_l\right) e^{-\delta k_l^2 \gamma^2} d(\delta k_l)}{\int_{-\infty}^{\infty} e^{-\delta k_l^2 \gamma^2} d(\delta k_l)}\right] = 2\left|\psi_A \psi_B\right|^2 \left[1 \pm e^{-(\mathbf{r}_{12})_l^2 / 4\gamma^2}\right]$$




Spatial correlation effects can be seen for $(\mathbf{r}_{12})_l \leq 2\gamma$.

(*ii*) The second half of this part is just geometry. Because we have a pulsed source, all the atoms are localized within a longitudinal distance $L$ at time $t = 0$. For two atoms in different parts of the cloud to reach the detector at the same later time $t = \tau$, it must be the case that the one starting further away had a larger enough velocity to 'catch-up' to the one starting closer. The largest difference in velocity between two particles arriving simultaneously at the detector will occur when one particle is from the "front" of the cloud, $v_{min} = d/\tau$, while the other is from the "back", $v_{max} = (d + L)/\tau$. Thus the difference in velocity, $\Delta v$, between any two particles detected at the same time must be $\leq L/\tau$. Using the relation $\hbar k = mv$, we find:

$$(\mathbf{k}_A - \mathbf{k}_B)_l \leq \frac{mL}{\hbar\tau} = \frac{mvL}{\hbar d},$$

written in terms of the "average" velocity $v = \frac{d}{\tau}$.

By the same arguments as given in part (b) above, in order to see second order correlation effects, we must have $\phi_l \equiv (\mathbf{k}_A - \mathbf{k}_B)_l (\mathbf{r}_{21})_l \ll 2\pi$. Since we have assumed that our detector has no longitudinal extent, $(\mathbf{r}_{21})_l = 0$ and this condition is trivially satisfied. In other words, all of the atoms detected at a particular time are guaranteed to be in the same longitudinal phase space cell. If, however, our detector has some finite response time, $t_r$, then we can attribute to it an 'effective' length $(\mathbf{r}_{21})_l = vt_r$. Now, in order to see second order correlations, we must have $(\mathbf{k}_A - \mathbf{k}_B)_l (\mathbf{r}_{21})_l \leq \frac{mvL}{\hbar d} vt_r \ll 2\pi$, or,

$$t_r \ll \frac{hd}{mLv^2} = 0.12 \text{ ms}$$

(d) Phase-Space Volume Enhancement

The initial phase space cell volume is $\delta x \delta y \delta z = \lambda_{dB}^3$.

In each dimension, we know that the uncertainty in position and momentum corresponding to a single phase space cell is given by $\delta k \delta r \simeq 2\pi$.

From part (b), in each of the two transverse dimensions we have $(\mathbf{k}_A - \mathbf{k}_B)_t = \delta k_t \simeq k_0 W/d$. Thus:

$$\delta x_t = \frac{2\pi}{\delta k_t} = \frac{d}{W}\frac{2\pi}{k_0} = \frac{d}{W}\lambda_{dB}.$$

From part (c), in the longitudinal direction we have $(\mathbf{k}_A - \mathbf{k}_B)_l = \delta k_l \simeq \frac{mvL}{\hbar d} = k_0 L/d$. Thus:

$$\delta x_l = \frac{2\pi}{\delta k_l} = \frac{d}{L}\frac{2\pi}{k_0} = \frac{d}{L}\lambda_{dB}.$$

Our new phase space cell volume (after expansion of the cloud) is $\delta x_t^2 \delta x_l = \frac{d^3}{W^2 L}\lambda_{dB}^3$. So the phase space volume has increased by $\frac{d^3}{W^2 L} \approx 10^{12}$ for $d = 10$cm and $L \approx W \approx 10\mu$m. The phase space density of the Lithium MOT is given by $n\lambda_{DB}^3$. Using the given numbers, the phase space density is $\sim 10^{-7}$, much lower than the point of degenearcy of $n\lambda_{DB}^3 \sim 1$.