Bose–Einstein Condensation
in Dilute Gases

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hyd. equations

Collective modes
Dynamics of the condensate

The time-dependent behaviour of Bose–Einstein condensed clouds, such as collective modes and the expansion of a cloud when released from a trap, is an important source of information about the physical nature of the condensate. In addition, the spectrum of elementary excitations of the condensate is an essential ingredient in calculations of thermodynamic properties. In this chapter we treat the dynamics of a condensate at zero temperature starting from a time-dependent generalization of the Gross–Pitaevskii equation used in Chapter 6 to describe static properties. From this equation one may derive equations very similar to those of classical hydrodynamics, which we shall use to calculate properties of collective modes.

We begin in Sec. 7.1 by describing the time-dependent Gross–Pitaevskii equation and deriving the hydrodynamic equations. We then use the hydrodynamic equations to determine the excitation spectrum of a homogeneous Bose gas (Sec. 7.2). Subsequently, we consider modes in trapped clouds (Sec. 7.3) within the hydrodynamic approach, and also describe the method of collective coordinates and the related variational method. In Sec. 7.4 we consider surface modes of oscillation, which resemble gravity waves on a liquid surface. The variational approach is used in Sec. 7.5 to treat the free expansion of a condensate upon release from a trap. Finally, in Sec. 7.6 we discuss solitons, which are exact one-dimensional solutions of the time-dependent Gross–Pitaevskii equation.

7.1 General formulation

In the previous chapter we saw that the equilibrium structure of the condensate is described by a time-independent Schrödinger equation with a non-linear contribution to the potential to take into account interactions between particles. To treat dynamical problems it is natural to use a time-dependent
generalization of this Schrödinger equation, with the same non-linear interaction term. This equation is the time-dependent Gross–Pitaevskii equation,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) + V(r)\psi(r, t) + U_0|\psi(r, t)|^2\psi(r, t) = \frac{i\hbar}{\partial t} \frac{\partial \psi(r, t)}{\partial t},$$  \hspace{1cm} (7.1)

and it is the basis for our discussion of the dynamics of the condensate.

The time-independent Gross–Pitaevskii equation, Eq. (6.11), is a non-linear Schrödinger equation with the chemical potential replacing the energy eigenvalue in the time-independent Schrödinger equation. To ensure consistency between the time-dependent Gross–Pitaevskii equation and the time-independent one, under stationary conditions $\psi(r, t)$ must develop in time as $\exp(-i\mu t/\hbar)$. The phase factor reflects the fact that microscopically $\psi$ is equal to the matrix element of the annihilation operator $\hat{\psi}$ between the ground state with $N$ particles and that with $N-1$ particles,

$$\psi(r, t) = \langle N-1 | \hat{\psi}(r) | N \rangle \propto \exp[-i(E_N - E_{N-1})t/\hbar],$$  \hspace{1cm} (7.2)

since the states $|N\rangle$ and $|N-1\rangle$ develop in time as $\exp(-iE_N t/\hbar)$ and $\exp(-iE_{N-1} t/\hbar)$. For large $N$ the difference in ground-state energies $E_N - E_{N-1}$ is equal to $\partial E/\partial N$, which is the chemical potential. Therefore this result is basically Josephson's relation for the development of the phase $\phi$ of the condensate wave function

$$\frac{d\phi}{dt} = -\frac{\mu}{\hbar}.$$  \hspace{1cm} (7.3)

Both for formal reasons as well as for applications a variational formulation analogous to that for static problems is useful. The time-dependent Gross–Pitaevskii equation (7.1) may be derived from the action principle

$$\delta \int_{t_1}^{t_2} L dt = 0,$$  \hspace{1cm} (7.4)

where the Lagrangian $L$ is given by

$$L = \int \frac{d^3x}{2} \frac{i\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - E$$

$$= \int \frac{d^3x}{2} \left[ \frac{i\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - E \right].$$  \hspace{1cm} (7.5)

Here $E$ is the energy, Eq. (6.9), and the energy density is given by

$$\mathcal{E} = \frac{\hbar^2}{2m} |\nabla \psi|^2 + V(r)|\psi|^2 + \frac{U_0}{2} |\psi|^4.$$  \hspace{1cm} (7.6)

In the variational principle (7.4) the variations of $\psi$ (or $\psi^*$) are arbitrary,
apart from the requirement that they vanish at \( t = t_1, \ t = t_2 \), and on any spatial boundaries for all \( t \). With a physically motivated choice of trial function for \( \psi \), this variational principle provides the foundation for approximate solutions of dynamical problems, as we shall illustrate in Sec. 7.3.3.

The physical content of the Gross–Pitaevskii equation (7.1) may be revealed by reformulating it as a pair of hydrodynamic equations, which we now derive.

### 7.1.1 The hydrodynamic equations

Under general, time-dependent conditions we may use instead of (7.1) an equivalent set of equations for the density, which is given by \( |\psi|^2 \), and the gradient of its phase, which is proportional to the local velocity of the condensate.

To understand the nature of the velocity of the condensate, we derive the continuity equation. If one multiplies the time-dependent Gross–Pitaevskii equation (7.1) by \( \psi^*(\mathbf{r}, t) \) and subtracts the complex conjugate of the resulting equation, one arrives at the equation

\[
\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left[ \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right] = 0. \tag{7.7}
\]

This is the same as one obtains from the usual (linear) Schrödinger equation, since the non-linear potential in the Gross–Pitaevskii equation is real. Equation (7.7) has the form of a continuity equation for the particle density, \( n = |\psi|^2 \), and it may be written as

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0, \tag{7.8}
\]

where the velocity of the condensate is defined by

\[
v = \frac{\hbar}{2mi} \frac{(\psi^* \nabla \psi - \psi \nabla \psi^*)}{|\psi|^2}. \tag{7.9}
\]

The momentum density \( \mathbf{j} \) is given by

\[
\mathbf{j} = \frac{\hbar}{2i} (\psi^* \nabla \psi - \psi \nabla \psi^*), \tag{7.10}
\]

and therefore the relation (7.9) is equivalent to the result

\[
\mathbf{j} = mnv, \tag{7.11}
\]

which states that the momentum density is equal to the particle mass times the particle current density.
Simple expressions for the density and velocity may be obtained if we write $\psi$ in terms of its amplitude $f$ and phase $\phi$,

$$\psi = fe^{i\phi},$$  \hspace{1cm} (7.12)

from which it follows that

$$n = f^2,$$  \hspace{1cm} (7.13)

and the velocity $\mathbf{v}$ is

$$\mathbf{v} = \frac{\hbar}{m} \nabla \phi.$$  \hspace{1cm} (7.14)

From Eq. (7.14) we conclude that the motion of the condensate corresponds to potential flow, since the velocity is the gradient of a scalar quantity, which is referred to as the velocity potential. For a condensate, Eq. (7.14) shows that the velocity potential is $\hbar \phi/m$. Provided that $\phi$ is not singular, we can immediately conclude that the motion of the condensate must be irrotational, that is

$$\nabla \times \mathbf{v} = \frac{\hbar}{m} \nabla \times \nabla \phi = 0.$$  \hspace{1cm} (7.15)

The possible motions of a condensate are thus much more restricted than those of a classical fluid.

The equations of motion for $f$ and $\phi$ may be found by inserting (7.12) into (7.1) and separating real and imaginary parts. Since

$$i \frac{\partial \psi}{\partial t} = i \frac{\partial f}{\partial t} e^{i\phi} - \frac{\partial \phi}{\partial t} fe^{i\phi}$$  \hspace{1cm} (7.16)

and

$$-\nabla^2 \psi = [-\nabla^2 f + (\nabla \phi)^2 f - i \nabla^2 \phi f - 2i \nabla \phi \cdot \nabla f] e^{i\phi},$$  \hspace{1cm} (7.17)

we obtain the two equations

$$\frac{\partial (f^2)}{\partial t} = -\frac{\hbar}{m} \nabla \cdot (f^2 \nabla \phi)$$  \hspace{1cm} (7.18)

and

$$-\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2mf} \nabla^2 f + \frac{1}{2} mv^2 + V(\mathbf{r}) + U_0f^2.$$  \hspace{1cm} (7.19)

Equation (7.18) is the continuity equation (7.8) expressed in the new

1 Note that this result applies only if $\phi$ is not singular. This condition is satisfied in the examples we consider in this chapter, but it fails at, e.g., the core of a vortex line. The properties of vortices will be treated in Chapter 9.
variables. To find the equation of motion for the velocity, given by Eq. (7.14), we take the gradient of Eq. (7.19), and the resulting equation is

\[
m \frac{\partial \mathbf{v}}{\partial t} = -\nabla (\tilde{\mu} + \frac{1}{2} m v^2),
\]  

(7.20)

where

\[
\tilde{\mu} = V + n U_0 - \frac{\hbar^2}{2 m \sqrt{n}} \nabla^2 \sqrt{n}.
\]  

(7.21)

Equation (7.19) may be expressed in terms of the functional derivative \(\delta E/\delta n\),

\[
\frac{\partial \phi(r, t)}{\partial t} = -\frac{1}{\hbar} \frac{\delta E}{\delta n(r)}.
\]  

(7.22)

The quantity \(\delta E/\delta n(r)\) is the energy required to add a particle at point \(r\), and therefore this result is the generalization of the Josephson relation (7.3) to systems not in their ground states. Under stationary conditions \(\tilde{\mu} + \frac{1}{2} m v^2\) is a constant, and if in addition the velocity is zero, that is \(\phi\) is independent of position, \(\tilde{\mu}\) is a constant, which is precisely the time-independent Gross–Pitaevskii equation (6.11).

The quantity \(n U_0\) in Eq. (7.21) is the expression for the chemical potential of a uniform Bose gas, omitting contributions from the external potential. At zero temperature, changes in the chemical potential for a bulk system are related to changes in the pressure \(p\) by the Gibbs–Duhem relation \(d\mu = nd\mu\), a result easily confirmed for the uniform dilute Bose gas, since \(\mu = n U_0\) and \(p = -\partial E/\partial V = n^2 U_0/2\) (see Eqs. (6.12) and (6.6), respectively). Equation (7.20) may therefore be rewritten in the form

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{mn} \nabla p - \nabla \left( \frac{v^2}{2} \right) + \frac{1}{m} \nabla \left( \frac{\hbar^2}{2 m \sqrt{n}} \nabla^2 \sqrt{n} \right) - \frac{1}{m} \nabla V.
\]  

(7.23)

Equations (7.8) and (7.23) are very similar to the hydrodynamic equations for a perfect fluid. If we denote the velocity of the fluid by \(\mathbf{v}\), the continuity equation (7.8) has precisely the same form as for a perfect fluid, while the analogue of Eq. (7.23) is the Euler equation

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{mn} \nabla p = -\frac{1}{m} \nabla \mathbf{V},
\]  

(7.24)

The functional derivative \(\delta E/\delta n(r)\) of the energy is defined according to the equation \(\delta E = \int d\mathbf{r} [\delta E/\delta n(r)] \delta n(r)\), and it is a function of \(r\) with the dimension of energy. It is given in terms of the energy density (7.6), which is a function of \(n\) and \(\nabla n\), by

\[
\frac{\delta E}{\delta n(r)} = \frac{\delta E}{\delta n} = \frac{\partial E}{\partial n} - \sum_i \frac{\partial}{\partial x_i} \frac{\delta E}{\delta (\partial n/\partial x_i)},
\]  

where the sum is over the three spatial coordinates.
or
\[
\frac{\partial v}{\partial t} - v \times (\nabla \times v) = -\frac{1}{mn} \nabla p - \nabla \left( \frac{v^2}{2} \right) - \frac{1}{m} \nabla V. \tag{7.25}
\]

Here the pressure \( p \) is that of the fluid, which generally has a form different from that of the condensate.

There are two differences between equations (7.23) and (7.25). The first is that the Euler equation contains the term \( v \times (\nabla \times v) \). However, since the velocity field of the superfluid corresponds to potential flow, \( \nabla \times v = 0 \), the term \( v \times (\nabla \times v) \) for such a flow would not contribute in the Euler equation. We shall comment further on this term in the context of vortex motion at the end of Sec. 9.4. The only difference between the two equations for potential flow is therefore the third term on the right hand side of Eq. (7.23), which is referred to as the quantum pressure term. This describes forces due to spatial variations in the magnitude of the wave function for the condensed state. Like the term \( \nabla v^2/2 \), its origin is the kinetic energy term \( \hbar^2 |\nabla \psi|^2/2m = mnv^2/2 + \hbar^2 (\nabla f)^2/2m \) in the energy density, but the two contributions correspond to different physical effects: the first is the kinetic energy of motion of particles, while the latter corresponds to 'zero-point motion', which does not give rise to particle currents. If the spatial scale of variations of the condensate wave function is \( l \), the pressure term in Eq. (7.23) is of order \( nU_0/ml \), while the quantum pressure term is of order \( \hbar^2/m^2l^3 \). Thus the quantum pressure term dominates the usual pressure term if spatial variations of the density occur on length scales \( l \) less than or of order the coherence length \( \xi \sim \hbar/(mnU_0)^{1/2} \) (see Eq. (6.62)), and it becomes less important on larger length scales.

As we have seen, motions of the condensate may be specified in terms of a local density and a local velocity. The reason for this is that the only degrees of freedom are those of the condensate wave function, which has a magnitude and a phase. Ordinary liquids and gases have many more degrees of freedom and, as a consequence, it is in general necessary to employ a microscopic description, e.g., in terms of the distribution function for the particles. However, a hydrodynamic description is possible for ordinary gases and liquids if collisions between particles are sufficiently frequent that thermodynamic equilibrium is established locally. The state of the fluid may then be specified completely in terms of the local particle density (or equivalently the mass density), the local velocity, and the local temperature. At zero temperature, the temperature is not a relevant variable, and the motion may be described in terms of the local density and the local fluid velocity, just as for a condensate. The reason that the equations of motion for a
condensate and for a perfect fluid are so similar is that they are expressions of the conservation laws for particle number and for total momentum. However, the physical reasons for a description in terms of a local density and a local velocity being possible are quite different for the two situations.

7.2 Elementary excitations

The properties of elementary excitations may be investigated by considering small deviations of the state of the gas from equilibrium and finding periodic solutions of the time-dependent Gross–Pitaevskii equation. An equivalent approach is to use the hydrodynamic formulation given above, and we begin by describing this. In Chapter 8 we shall consider the problem on the basis of microscopic theory. We write the density as \( n = n_{\text{eq}} + \delta n \), where \( n_{\text{eq}} \) is the equilibrium density and \( \delta n \) the departure of the density from its equilibrium value. On linearizing Eqs. (7.8), (7.20), and (7.21) by treating the velocity \( \mathbf{v} \) and \( \delta n \) as small quantities, one finds

\[
\frac{\partial \delta n}{\partial t} = -\nabla \cdot (n_{\text{eq}} \mathbf{v})
\]  

(7.26)

and

\[
m \frac{\partial \mathbf{v}}{\partial t} = -\nabla \delta \mu,
\]  

(7.27)

where \( \delta \mu \) is obtained by linearizing (7.21). Taking the time derivative of (7.26) and eliminating the velocity by means of (7.27) results in the equation of motion

\[
m \frac{\partial^2 \delta n}{\partial t^2} = \nabla \cdot (n_{\text{eq}} \nabla \delta \mu).
\]  

(7.28)

This equation describes the excitations of a Bose gas in an arbitrary potential. To keep the notation simple, we shall henceforth in this chapter denote the equilibrium density by \( n \). Note that \( n \) is the density of the condensate, since we neglect the zero-temperature depletion of the condensate. In Chapter 10, which treats the dynamics at finite temperature, we shall denote the condensate density by \( n_0 \), in order to distinguish it from the total density, which includes a contribution from thermal excitations.

A uniform gas

As a first example we investigate the spectrum for a homogeneous gas, where the external potential \( V \) is constant. In the undisturbed state the density \( n \) is the same everywhere and it may therefore be taken outside the spatial derivatives. We look for travelling-wave solutions, proportional
to $\exp(iq \cdot r - i\omega t)$, where $q$ is the wave vector and $\omega$ the frequency. From Eq. (7.21) the change in $\tilde{\mu}$ is seen to be equal to

$$\delta \tilde{\mu} = \left( U_0 + \frac{\hbar^2 q^2}{4mn} \right) \delta n$$

(7.29)

and the equation of motion becomes

$$m\omega^2 \delta n = \left( nU_0q^2 + \frac{\hbar^2 q^4}{4m} \right) \delta n.$$  

(7.30)

To make contact with the microscopic calculations to be described later, it is convenient to work with the energy of an excitation, $\epsilon_q$, rather than the frequency. Non-vanishing solutions of (7.30) are possible only if the frequency is given by $\omega = \pm \epsilon_q / \hbar$, where

$$\epsilon_q = \sqrt{2nU_0\epsilon_q^0 + (\epsilon_q^0)^2}.$$  

(7.31)

Here

$$\epsilon_q^0 = \frac{\hbar^2 q^2}{2m}$$  

(7.32)

is the free-particle energy. This spectrum was first derived by Bogoliubov from microscopic theory [1]. In the following discussion we shall adopt the convention that the branch of the square root to be used is the positive one.

The excitation spectrum (7.31) is plotted in Fig. 7.1. For small $q$, $\epsilon_q$ is a linear function of $q$,

$$\epsilon_q \simeq s \hbar q,$$  

(7.33)

and the spectrum is sound-like. The velocity $s$ is seen to be

$$s = \sqrt{nU_0/m}.$$  

(7.34)

This result agrees with the expression for the sound velocity calculated from the hydrodynamic result $s^2 = dp/d\rho = (n/m)d\mu/dn$, where $\rho = nm$ is the mass density. The repulsive interaction has thus turned the energy spectrum at long wavelengths, which is quadratic in $q$ for free particles, into a linear one, in agreement with what is observed experimentally in liquid $^4$He. As we shall see in Chapter 10, the linear spectrum at long wavelengths provides the key to superfluid behaviour, and it was one of the triumphs of Bogoliubov's pioneering calculation. In the hydrodynamic description the result is almost 'obvious', since sound waves are well-established excitations of hydrodynamic systems. What is perhaps surprising is that at short wavelengths the leading contributions to the spectrum are

$$\epsilon_q \simeq \epsilon_q^0 + nU_0,$$  

(7.35)
Fig. 7.1. Excitation spectrum of a homogeneous Bose gas (full line) plotted as a function of the wave number expressed as the dimensionless variable $\frac{\hbar q}{ms}$, where the sound velocity $s$ is given by Eq. (7.34). The expansion (7.35) for high wave number is shown as a dotted line.

which is also shown in Fig. 7.1. This is the free-particle spectrum plus a mean-field contribution. The transition between the linear spectrum and the quadratic one occurs when the kinetic energy, $\frac{\hbar^2 q^2}{2m}$, becomes large compared with the potential energy of a particle $\sim nU_0$, or in other words the 'quantum pressure' term dominates the usual pressure term. This occurs at a wave number $\sim (2mnU_0)^{1/2}/\hbar$, which is the inverse of the coherence length, $\xi$, Eq. (6.62). The coherence length is related to the sound velocity, Eq. (7.34) by $\xi = \hbar/\sqrt{2m}$. On length scales longer than $\xi$, atoms move collectively, while on shorter length scales, they behave as free particles. The spectrum of elementary excitations in superfluid liquid $^4$He differs from that for a dilute gas because of the strong short-range correlations. The first satisfactory account of the roton part of the spectrum was given by Feynman [2].

As a generalization of the above approach one may calculate the response of the condensate to a space- and time-dependent external potential $V(r,t) = V_q \exp(iq \cdot r - i\omega t)$. There is then an additional term $V_q$ in the equation for $\delta \mu$, and one finds

$$m \left( \omega^2 - \frac{\xi^2}{\hbar^2} \right) \delta n = nq^2V_q,$$  \hfill (7.36)
\[ \delta n = \chi(q, \omega)V_q, \]  
\[ \chi(q, \omega) = \frac{n q^2}{m(\omega^2 - q^2 / \hbar^2)} \] 

is the density–density response function for the condensate. Thus the response diverges if the frequency of the external potential is equal to the frequency of an elementary excitation of the condensate. In the final chapter we shall use the expression for the response function to calculate how the interaction between two fermions is affected by the presence of a condensate of bosons.

**The Bogoliubov equations**

An alternative route to calculating the excitation spectrum is to start from the Gross–Pitaevskii equation directly, without introducing the hydrodynamic variables. This approach complements the hydrodynamic one since it emphasizes single-particle behaviour and shows how the collective effects at long wavelengths come about. Let us denote the change in \( \psi \) by \( \delta \psi \). Linearizing the Gross–Pitaevskii equation (7.1), one finds

\[ -\frac{\hbar^2}{2m} \nabla^2 \delta \psi(r, t) + V(r) \delta \psi(r, t) + U_0[2|\psi(r, t)|^2 \delta \psi(r, t) + \psi(r, t)^2 \delta \psi^*(r, t)] = i\hbar \frac{\partial \delta \psi(r, t)}{\partial t} \]  

and

\[ -\frac{\hbar^2}{2m} \nabla^2 \delta \psi^*(r, t) + V(r) \delta \psi^*(r, t) + U_0[2|\psi(r, t)|^2 \delta \psi^*(r, t) + \psi^*(r, t)^2 \delta \psi(r, t)] = -i\hbar \frac{\partial \delta \psi^*(r, t)}{\partial t}. \]  

Here \( \psi(r, t) \) is understood to be the condensate wave function in the unperturbed state, which we may write as \( \psi = \sqrt{n(r)} e^{-i\mu t / \hbar} \), where \( n(r) \) is the equilibrium density of particles and \( \mu \) is the chemical potential of the unperturbed system. To avoid carrying an arbitrary phase factor along in our calculations we have taken the phase of the condensate wave function at \( t = 0 \) to be zero. We wish to find solutions of these equations which are periodic in time, apart from the overall phase factor \( e^{-i\mu t / \hbar} \) present for the unperturbed state. We therefore search for solutions of the form

\[ \delta \psi(r, t) = e^{-i\mu t / \hbar} [u(r)e^{-i\omega t} - v^*(r)e^{i\omega t}], \]
where $u(r)$ and $v(r)$ are functions to be determined. The overall phase factor $e^{-iut/h}$ is necessary to cancel the effects of the phases of $\psi(r,t)^2$ and $\psi^*(r,t)^2$ in Eqs. (7.39) and (7.40), and thereby ensure that the equations can be satisfied for all time. The choice of the sign in front of $v$ is a matter of convention, and we take it to be negative so that $u$ and $v$ will have the same sign. Since the equations couple $\delta \psi$ and $\delta \psi^*$, they cannot be satisfied unless both positive and negative frequency components are allowed for. By inserting the ansatz (7.41) into the two equations (7.39) and (7.40) we obtain the following pair of coupled equations for $u(r)$ and $v(r)$:

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + 2n(r)U_0 - \mu - \hbar \omega \right] u(r) - n(r)U_0 v(r) = 0 \quad (7.42)$$

and

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + 2n(r)U_0 - \mu + \hbar \omega \right] v(r) - n(r)U_0 u(r) = 0, \quad (7.43)$$

which are referred to as the Bogoliubov equations.

We now apply this formalism to the uniform Bose gas, $V(r) = 0$. Because of the translational invariance the solutions may be chosen to be of the form

$$u(r) = u_q \frac{e^{iq \cdot r}}{V^{1/2}} \quad \text{and} \quad v(r) = v_q \frac{e^{iq \cdot r}}{V^{1/2}}, \quad (7.44)$$

where we have introduced the conventional normalization factor $1/V^{1/2}$ explicitly, $V$ being the volume of the system.

The chemical potential for the uniform system is given by $nU_0$ (Eq. (6.12)), and thus the Bogoliubov equations are

$$\left( \frac{\hbar^2 q^2}{2m} + nU_0 - \hbar \omega \right) u_q - nU_0 v_q = 0 \quad (7.45)$$

and

$$\left( \frac{\hbar^2 q^2}{2m} + nU_0 + \hbar \omega \right) v_q - nU_0 u_q = 0. \quad (7.46)$$

The two equations are consistent only if the determinant of the coefficients vanishes. With the definition (7.32) this leads to the condition

$$(\epsilon_q^0 + nU_0 - \hbar \omega)(\epsilon_q^0 + nU_0 - \hbar \omega) - n^2 U_0^2 = 0, \quad (7.47)$$

or

$$(\hbar \omega)^2 = (\epsilon_q^0 + nU_0)^2 - (nU_0)^2 = \epsilon_q^0(\epsilon_q^0 + 2nU_0), \quad (7.48)$$

which agrees with the spectrum (7.31) obtained earlier from the hydrodynamic approach.
The nature of the excitations may be elucidated by investigating the behaviour of the coefficients \( u_q \) and \( v_q \). Here we shall consider the case of repulsive interactions. For the positive energy solutions, one has

\[
\begin{align*}
v_q &= \frac{nU_0}{\epsilon_q + \xi_q} u_q, \\
\xi_q &= \epsilon_q^0 + nU_0
\end{align*}
\]

(7.49)

(7.50)

where

is the energy of an excitation if one neglects coupling between \( u_q \) and \( v_q \). The normalization of \( u_q \) and \( v_q \) is arbitrary, but as we shall see from the quantum-mechanical treatment in Sec. 8.1, a convenient one is

\[
|u_q|^2 - |v_q|^2 = 1,
\]

(7.51)

since this ensures that the new operators introduced there satisfy Bose commutation relations. The Bogoliubov equations are unaltered if \( u_q \) and \( v_q \) are multiplied by an arbitrary phase factor. Therefore, without loss of generality we may take \( u_q \) and \( v_q \) to be real. With this choice one finds that

\[
u^2_q = \frac{1}{2} \left( \frac{\xi_q}{\epsilon_q} + 1 \right)
\]

(7.52)

and

\[
v^2_q = \frac{1}{2} \left( \frac{\xi_q}{\epsilon_q} - 1 \right).
\]

(7.53)

In terms of \( \xi_q \), the excitation energy is given by

\[
\epsilon_q = \sqrt{\xi_q^2 - (nU_0)^2}.
\]

(7.54)

The coefficients \( u_q \) and \( v_q \) are exhibited as functions of the dimensionless variable \( \hbar q/\mu s \) in Fig. 7.2.

For the positive energy solution, \( v_q \) tends to zero as \( 1/q^2 \) for large \( q \), and in this limit \( \delta \psi = e^{i(q \cdot r - \omega_q t)} / V^{1/2} \), with \( \omega_q = \epsilon_q / \hbar \). This corresponds to addition of a single particle with momentum \( \hbar q \), and the removal of a particle in the zero-momentum state, as will be made explicit when the quantum-mechanical theory is presented in Chapter 8. At smaller momenta, excitations are linear superpositions of the state in which a particle with momentum \( \hbar q \) is added (and a particle in the condensate is removed) and the state in which a particle with momentum \(-\hbar q \) is removed (and a particle added to the condensate). At long wavelengths \( u_q \) and \( v_q \) diverge as \( 1/q^{1/2} \), and the two components of the wave function are essentially equal in magnitude.
7.2 Elementary excitations

Fig. 7.2. The coefficients $u_q$ and $v_q$ given by Eqs. (7.52)–(7.53) as functions of the wave number, expressed as the dimensionless variable $\hbar q/ms$.

The algebraic expressions for the excitation spectrum and the factors $u_q$ and $v_q$ are completely analogous to those for a superconductor in the Bardeen–Cooper–Schrieffer (BCS) theory, apart from some sign changes due to the fact that we are here dealing with bosons rather than fermions. In the BCS theory, which we shall describe in Sec. 14.3 in the context of the transition to a superfluid state for dilute Fermi gases, the dispersion relation for an elementary excitation is $\epsilon_q = \sqrt{\xi_q^2 + \Delta^2}$, where $\xi_q$ is the normal-state energy of a particle measured with respect to the chemical potential, as for the boson problem we consider, and $\Delta$ is the superconducting energy gap. Thus one sees that for bosons the excitation energy is obtained by replacing $\Delta^2$ in the BCS expression by $-(nU_0)^2$.

**Attractive interactions**

If the interaction is attractive, the sound speed is imaginary, which indicates that long-wavelength modes grow or decay exponentially in time, rather than oscillate. This signals an instability of the system due to the attractive interaction tending to make atoms clump together. However at shorter wavelengths modes are stable, since the free-particle kinetic energy term dominates in the dispersion relation. The lowest wave number $q_c$ for which
the mode is stable is given by the condition that its frequency vanish. Thus from Eq. (7.31)
\[ \epsilon_{q_c}^0 + 2nU_0 = 0 \]  
(7.55)
or
\[ q_c^2 = -\frac{4mnU_0}{\hbar^2} = 16\pi n|a|, \]  
(7.56)
where we have used Eq. (5.39) to express \( U_0 \) in terms of the scattering length \( a \). This shows that the spatial scale of unstable modes is greater than or of order the coherence length, Eq. (6.62), evaluated using the absolute magnitude of the scattering length.

It is instructive to relate these ideas for bulk matter to a cloud in a trap. For simplicity, we consider the trap to be a spherical container with radius \( R_0 \). The lowest mode has a wave number of order \( 1/R_0 \), and the density is \( n \sim N/R_0^3 \). Thus according to (7.56) the lowest mode is stable if the number of particles is less than a critical value \( N_c \) given by \( 1/R_0^2 \sim N_c|a|/R_0^3 \) or
\[ N_c \sim \frac{R_0}{|a|}, \]  
(7.57)
and unstable for larger numbers of particles. The physics of the instability is precisely the same as that considered in Chapter 6 in connection with the energy of a cloud: for sufficiently large numbers of particles, the zero-point energy of atoms is too small to overcome the attraction between them. In the present formulation, the zero-point energy is the kinetic energy of the lowest mode in the well. To make contact with the calculations for a harmonic-oscillator trap in Chapter 6, we note that the estimate (7.57) is consistent with the earlier result (6.29) if the radius \( R_0 \) of the container is replaced by the oscillator length.

### 7.3 Collective modes in traps
Calculating the properties of modes in a homogeneous gas is relatively straightforward because there are only two length scales in the problem, the coherence length and the wavelength of the excitation. For a gas in a trap there is an additional length, the spatial extent of the cloud, and moreover the coherence length varies in space. However, we have seen in Chapter 6 that static properties of clouds may be calculated rather precisely if the number of atoms is sufficiently large, \( Na/\bar{a} \gg 1 \), since under these conditions the kinetic energy associated with the confinement of atoms within the cloud, which gives rise to the quantum pressure, may be neglected. It is
therefore of interest to explore the properties of modes when the quantum pressure term in the equation of motion is neglected. For such an approximation to be reliable, a mode must not be concentrated in the boundary layer of thickness \( \sim \delta \), and must vary in space only on length scales large compared with the local coherence length. In this approach one can describe collective modes, but not excitations which are free-particle-like.

The basic equation for linear modes was derived earlier in Eq. (7.28). When the quantum pressure term is neglected, the quantity \( \bar{\mu} \) reduces to \( nU_0 + V \), and therefore

\[
\delta \bar{\mu} = U_0 \delta n. \tag{7.58}
\]

Inserting this result into Eq. (7.28), we find that the density disturbance satisfies the equation

\[
m \frac{\partial^2 \delta n}{\partial t^2} = U_0 \nabla \cdot (n \nabla \delta n). \tag{7.59}
\]

If we consider oscillations with time dependence \( \delta n \propto e^{-i\omega t} \), the differential equation (7.59) simplifies to

\[
-\omega^2 \delta n = \frac{U_0}{m} (\nabla n \cdot \nabla \delta n + n \nabla^2 \delta n). \tag{7.60}
\]

The equilibrium density is given by

\[
n = \frac{\mu - V(r)}{U_0}, \tag{7.61}
\]

and therefore the equation (7.60) reduces to

\[
\omega^2 \delta n = \frac{1}{m} \{ \nabla V \cdot \nabla \delta n - [\mu - V(r)] \nabla^2 \delta n \}. \tag{7.62}
\]

In the following two subsections we discuss solutions to (7.62) and the associated mode frequencies [3].

### 7.3.1 Traps with spherical symmetry

First we consider an isotropic harmonic trap (\( \lambda = 1 \)). The potential is

\[
V(r) = \frac{1}{2} m\omega_0^2 r^2, \tag{7.63}
\]

and in the Thomas–Fermi approximation the chemical potential and the radius of the cloud are related by the equation \( \mu = m\omega_0^2 R^2 / 2 \). It is natural