

Bose-Einstein Condensation in Dilute Gases



C. J. Pethick
Nordita

H. Smith
University of Copenhagen

pp 205 - 214

Bogoliubov transformation

 **CAMBRIDGE**
UNIVERSITY PRESS

classical theory of electrodynamics, in which a state is characterized by classical electric and magnetic fields, rather than by creation and annihilation operators for photons.

To take into account quantum fluctuations about the state in which all atoms are condensed in a single quantum state it is natural to write¹

$$\hat{\psi}(\mathbf{r}) = \psi(\mathbf{r}) + \delta\hat{\psi}(\mathbf{r}). \quad (8.2)$$

If the fluctuation term $\delta\hat{\psi}(\mathbf{r})$ is neglected, the Hamiltonian is equivalent to the energy expression which leads to the Gross-Pitaevskii equation.

8.1 Excitations in a uniform gas

As a first illustration we consider a uniform gas of interacting bosons contained in a box of volume V . The Hamiltonian (8.1) then becomes

$$H = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}}^0 a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{U_0}{2V} \sum_{\mathbf{p}, \mathbf{p}', \mathbf{q}} a_{\mathbf{p}+\mathbf{q}}^{\dagger} a_{\mathbf{p}'-\mathbf{q}}^{\dagger} a_{\mathbf{p}'} a_{\mathbf{p}}, \quad (8.3)$$

where $\epsilon_{\mathbf{p}}^0 = p^2/2m$. Here the operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\dagger}$ that destroy and create bosons in the state with momentum \mathbf{p} satisfy the usual Bose commutation relations

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}] = \delta_{\mathbf{p}, \mathbf{p}'}, \quad [a_{\mathbf{p}}, a_{\mathbf{p}'}] = 0, \quad \text{and} \quad [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}'}^{\dagger}] = 0. \quad (8.4)$$

We assume that in the interacting system the lowest-lying single-particle state is macroscopically occupied, that is N_0/N tends to a non-zero value in the thermodynamic limit when N and V tend to infinity in such a way that the density N/V remains constant. In the unperturbed system we have

$$a_0^{\dagger}|N_0\rangle = \sqrt{N_0+1}|N_0+1\rangle \quad \text{and} \quad a_0|N_0\rangle = \sqrt{N_0}|N_0-1\rangle, \quad (8.5)$$

and in the Hamiltonian we therefore replace a_0 and a_0^{\dagger} by $\sqrt{N_0}$, as was first done by Bogoliubov [1]. This is equivalent to using Eq. (8.2) with the wave function for the condensed state given by $\psi = \sqrt{N_0}\phi_0$, where $\phi_0 = V^{-1/2}$ is the wave function for the zero-momentum state.

Within the Bogoliubov approach one assumes that $\delta\hat{\psi}(\mathbf{r})$ is small and retains in the interaction all terms which have (at least) two powers of $\psi(\mathbf{r})$ or $\psi^*(\mathbf{r})$. This is equivalent to including terms which are no more than

¹ In this chapter we use the notation $\hat{\psi}$ to distinguish the annihilation operator from the wave function ψ . When ambiguities do not exist (as with the annihilation operator $a_{\mathbf{p}}$) we omit the 'hat'.

quadratic in $\delta\hat{\psi}(\mathbf{r})$ and $\delta\hat{\psi}^\dagger(\mathbf{r})$, that is in $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ for $\mathbf{p} \neq 0$. One finds

$$H = \frac{N_0^2 U_0}{2V} + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} (\epsilon_{\mathbf{p}}^0 + 2n_0 U_0) a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{n_0 U_0}{2} \sum_{\mathbf{p}(\mathbf{p} \neq 0)} (a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger + a_{\mathbf{p}} a_{-\mathbf{p}}), \quad (8.6)$$

where $n_0 = N_0/V$ is the density of particles in the zero-momentum state.² The first term is the energy of N_0 particles in the zero-momentum state, and the second is that of independent excitations with energy $\epsilon_{\mathbf{p}}^0 + 2n_0 U_0$, which is the energy of an excitation moving in the Hartree–Fock mean field produced by interactions with other atoms. To see this it is convenient to consider an interaction $U(r)$ with non-zero range instead of the contact one, and introduce its Fourier transform $U(p)$ by

$$U(p) = \int d\mathbf{r} U(r) \exp(-i\mathbf{p} \cdot \mathbf{r} / \hbar). \quad (8.7)$$

When the operators a_0 and a_0^\dagger in the Hamiltonian are replaced by c numbers, the term in the interaction proportional to N_0 is

$$\sum_{\mathbf{p}(\mathbf{p} \neq 0)} n_0 [U(0) + U(p)] a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}(\mathbf{p} \neq 0)} n_0 U(p) (a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger + a_{\mathbf{p}} a_{-\mathbf{p}}). \quad (8.8)$$

The $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ term has two contributions. The first, $n_0 U(0)$, is the Hartree energy, which comes from the direct interaction of a particle in the state \mathbf{p} with the N_0 atoms in the zero-momentum state. The second is the exchange, or Fock, term, in which an atom in the state \mathbf{p} is scattered into the zero-momentum state, while a second atom is simultaneously scattered from the condensate to the state \mathbf{p} . These identifications will be further elucidated in Sec. 8.3.1 below where we consider the Hartree–Fock approximation in greater detail. For a contact interaction the Fourier transform of the interaction $U(p)$ is independent of p , and therefore the Hartree and Fock terms are both equal to $n_0 U_0$. The final terms in Eqs. (8.6) and (8.8) correspond to the scattering of two atoms in the condensate to states with momenta $\pm \mathbf{p}$ and the inverse process in which two atoms with momenta $\pm \mathbf{p}$ are scattered into the condensate.

The task now is to find the eigenvalues of the Hamiltonian (8.6). The original Hamiltonian conserved the number of particles, and therefore we wish to find the eigenvalues of the new Hamiltonian for a fixed average

² In this chapter and the following ones it is important to distinguish between the condensate density and the total density, and we shall denote the condensate density by n_0 and the total density by n .

particle number. The operator for the total particle number is given by

$$\hat{N} = \sum_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}, \quad (8.9)$$

which on treating the zero-momentum-state operators as c numbers becomes

$$\hat{N} = N_0 + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}. \quad (8.10)$$

Expressed in terms of the total number of particles, the Hamiltonian (8.6) may be written

$$H = \frac{N^2 U_0}{2V} + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} \left[(\epsilon_{\mathbf{p}}^0 + n_0 U_0) a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{n_0 U_0}{2} (a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} + a_{\mathbf{p}} a_{-\mathbf{p}}) \right], \quad (8.11)$$

where in the first term we have replaced \hat{N} by its expectation value. This is permissible since the fluctuation in the particle number is small. Since we consider states differing little from the state with all particles in the condensed state it makes no difference whether the condensate density or the total density appears in the terms in the sum. The reduction of the coefficient of $a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}$ from $\epsilon_{\mathbf{p}}^0 + 2n_0 U_0$ to $\epsilon_{\mathbf{p}}^0 + n_0 U_0$, is due to the condition that the total number of particles be fixed. In the classical treatment of excitations in Chapter 7 this corresponds to the subtraction of the chemical potential, since for the uniform Bose gas at zero temperature, the chemical potential is $n_0 U_0$, Eq. (6.12).

The energy $\epsilon_{\mathbf{p}}^0 + n_0 U_0$ does not depend on the direction of \mathbf{p} , and therefore we may write the Hamiltonian (8.11) in the symmetrical form

$$H = \frac{N^2 U_0}{2V} + \sum'_{\mathbf{p}(\mathbf{p} \neq 0)} [(\epsilon_{\mathbf{p}}^0 + n_0 U_0)(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} a_{-\mathbf{p}}) + n_0 U_0(a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} + a_{\mathbf{p}} a_{-\mathbf{p}})], \quad (8.12)$$

where the prime on the sum indicates that it is to be taken only over one half of momentum space, since the terms corresponding to \mathbf{p} and $-\mathbf{p}$ must be counted only once.

8.1.1 The Bogoliubov transformation

The structure of the Hamiltonian is now simple, since it consists of a sum of independent terms of the form

$$\epsilon_0(a^{\dagger} a + b^{\dagger} b) + \epsilon_1(a^{\dagger} b^{\dagger} + b a). \quad (8.13)$$

Here ϵ_0 and ϵ_1 are c numbers. The operators a^\dagger and a create and annihilate bosons in the state with momentum \mathbf{p} , and b^\dagger and b are the corresponding operators for the state with momentum $-\mathbf{p}$.

The eigenvalues and eigenstates of this Hamiltonian may be obtained by performing a canonical transformation, as Bogoliubov did in the context of liquid helium [1]. This method has proved to be very fruitful, and it is used extensively in the theory of superconductivity and of magnetism as well as in other fields. We shall use it again in Chapter 14 when we consider pairing of fermions. The basic idea is to introduce a new set of operators α and β such that the Hamiltonian has only terms proportional to $\alpha^\dagger\alpha$ and $\beta^\dagger\beta$.

Creation and annihilation operators for bosons obey the commutation relations

$$[a, a^\dagger] = [b, b^\dagger] = 1, \quad \text{and} \quad [a, b^\dagger] = [b, a^\dagger] = 0. \quad (8.14)$$

We introduce new operators α and β by the transformation

$$\alpha = ua + vb^\dagger, \quad \beta = ub + va^\dagger, \quad (8.15)$$

where u and v are coefficients to be determined. We require that also these operators satisfy Bose commutation rules,

$$[\alpha, \alpha^\dagger] = [\beta, \beta^\dagger] = 1, \quad [\alpha, \beta^\dagger] = [\beta, \alpha^\dagger] = 0. \quad (8.16)$$

Since the phases of u and v are arbitrary, we may take u and v to be real. By inserting (8.15) into (8.16) and using (8.14) one sees that u and v must satisfy the condition

$$u^2 - v^2 = 1. \quad (8.17)$$

The inverse transformation corresponding to (8.15) is

$$a = u\alpha - v\beta^\dagger, \quad b = u\beta - v\alpha^\dagger. \quad (8.18)$$

We now substitute (8.18) in (8.13) and obtain the result

$$H = 2v^2\epsilon_0 - 2uv\epsilon_1 + [\epsilon_0(u^2 + v^2) - 2uv\epsilon_1](\alpha^\dagger\alpha + \beta^\dagger\beta) + [\epsilon_1(u^2 + v^2) - 2uv\epsilon_0](\alpha\beta + \beta^\dagger\alpha^\dagger). \quad (8.19)$$

The term proportional to $\alpha\beta + \beta^\dagger\alpha^\dagger$ can be made to vanish by choosing u and v so that its coefficient is zero:

$$\epsilon_1(u^2 + v^2) - 2uv\epsilon_0 = 0. \quad (8.20)$$

The sign of u is arbitrary, and if we adopt the convention that it is positive,

the normalization condition (8.17) is satisfied by the following parametrization of u and v ,

$$u = \cosh t, \quad v = \sinh t, \quad (8.21)$$

which in turn implies that the condition (8.20) may be written as

$$\epsilon_1(\cosh^2 t + \sinh^2 t) - 2\epsilon_0 \sinh t \cosh t = 0, \quad (8.22)$$

or

$$\tanh 2t = \frac{\epsilon_1}{\epsilon_0}. \quad (8.23)$$

From this result one finds

$$u^2 = \frac{1}{2} \left(\frac{\epsilon_0}{\epsilon} + 1 \right) \quad \text{and} \quad v^2 = \frac{1}{2} \left(\frac{\epsilon_0}{\epsilon} - 1 \right), \quad (8.24)$$

where

$$\epsilon = \sqrt{\epsilon_0^2 - \epsilon_1^2}. \quad (8.25)$$

It is necessary to choose the positive branch of the square root, since otherwise u and v would be imaginary, contrary to our initial assumption. Solving for $u^2 + v^2$ and $2uv$ in terms of the ratio ϵ_1/ϵ_0 and inserting the expressions into (8.19) leads to the result

$$H = \epsilon(\alpha^\dagger \alpha + \beta^\dagger \beta) + \epsilon - \epsilon_0. \quad (8.26)$$

The ground-state energy is $\epsilon - \epsilon_0$, which is negative, and the excited states correspond to the addition of two independent kinds of bosons with energy ϵ , created by the operators α^\dagger and β^\dagger . For ϵ to be real, the magnitude of ϵ_0 must exceed that of ϵ_1 . If $|\epsilon_1| > |\epsilon_0|$, the excitation energy is imaginary, corresponding to an instability of the system.

8.1.2 Elementary excitations

We may now use the results of the previous subsection to bring the Hamiltonian (8.12) into diagonal form. We make the transformation

$$a_{\mathbf{p}} = u_{\mathbf{p}} \alpha_{\mathbf{p}} - v_{\mathbf{p}} \alpha_{-\mathbf{p}}^\dagger, \quad a_{-\mathbf{p}} = u_{\mathbf{p}} \alpha_{-\mathbf{p}} - v_{\mathbf{p}} \alpha_{\mathbf{p}}^\dagger, \quad (8.27)$$

where $a_{\mathbf{p}}$ corresponds to a in the simple model, $a_{-\mathbf{p}}$ to b , $\alpha_{\mathbf{p}}$ to α , and $\alpha_{-\mathbf{p}}$ to β . The result is

$$H = \frac{N^2 U_0}{2V} + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} \epsilon_{\mathbf{p}} \alpha_{\mathbf{p}}^\dagger \alpha_{\mathbf{p}} - \frac{1}{2} \sum_{\mathbf{p}(\mathbf{p} \neq 0)} (\epsilon_{\mathbf{p}}^0 + n_0 U_0 - \epsilon_{\mathbf{p}}) \quad (8.28)$$

with

$$\epsilon_p = \sqrt{(\epsilon_p^0 + n_0 U_0)^2 - (n_0 U_0)^2} = \sqrt{(\epsilon_p^0)^2 + 2\epsilon_p^0 n_0 U_0}. \quad (8.29)$$

The energy spectrum (8.29) agrees precisely with the result (7.48) derived in the previous chapter. For small p the energy is $\epsilon_p = sp$, where

$$s^2 = \frac{n_0 U_0}{m}. \quad (8.30)$$

The operators that create and destroy elementary excitations are given by

$$\alpha_{\mathbf{p}}^\dagger = u_p a_{\mathbf{p}}^\dagger + v_p a_{-\mathbf{p}}. \quad (8.31)$$

The coefficients satisfy the normalization condition

$$u_p^2 - v_p^2 = 1 \quad (8.32)$$

corresponding to Eq. (8.17) and are given explicitly by

$$u_p^2 = \frac{1}{2} \left(\frac{\xi_p}{\epsilon_p} + 1 \right) \quad \text{and} \quad v_p^2 = \frac{1}{2} \left(\frac{\xi_p}{\epsilon_p} - 1 \right), \quad (8.33)$$

where $\xi_p = \epsilon_p^0 + n_0 U_0$ is the difference between the Hartree-Fock energy of a particle and the chemical potential, Eq. (6.12).

Thus the system behaves as a collection of non-interacting bosons with energies given by the Bogoliubov spectrum previously derived from classical considerations in Chapter 7. In the ground state of the system there are no excitations, and thus $\alpha_{\mathbf{p}}|0\rangle = 0$.

Depletion of the condensate

The particle number is given by Eq. (8.10) which, rewritten in terms of $\alpha_{\mathbf{p}}^\dagger$ and $\alpha_{\mathbf{p}}$, has the form

$$\begin{aligned} \hat{N} = & N_0 + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} v_p^2 + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} (u_p^2 + v_p^2) \alpha_{\mathbf{p}}^\dagger \alpha_{\mathbf{p}} \\ & - \sum_{\mathbf{p}(\mathbf{p} \neq 0)} u_p v_p (\alpha_{\mathbf{p}}^\dagger \alpha_{-\mathbf{p}}^\dagger + \alpha_{-\mathbf{p}} \alpha_{\mathbf{p}}). \end{aligned} \quad (8.34)$$

In deriving this expression we used the Bose commutation relations to reorder operators so that the expectation value of the operator terms gives zero in the ground state. The physical interpretation of this expression is that the first term is the number of atoms in the condensate. The second term represents the depletion of the condensate by interactions when no real excitations are present. In the ground state of the interacting gas, not all

particles are in the zero-momentum state because the two-body interaction mixes into the ground-state components with atoms in other states. Consequently, the probability of an atom being in the zero-momentum state is reduced. The last terms correspond to the depletion of the condensate due to the presence of real excitations. For states which contain a definite number of elementary excitations, the expectation value of $\alpha_{\mathbf{p}}^\dagger \alpha_{-\mathbf{p}}^\dagger$ and its Hermitian conjugate vanish, and therefore the number operator may equivalently be written as

$$\hat{N} = N_0 + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} v_p^2 + \sum_{\mathbf{p}(\mathbf{p} \neq 0)} (u_p^2 + v_p^2) \alpha_{\mathbf{p}}^\dagger \alpha_{\mathbf{p}}. \quad (8.35)$$

This shows that when an excitation with non-zero momentum \mathbf{p} is added to the gas, keeping N_0 fixed, the number of particles changes by an amount

$$\nu_p = u_p^2 + v_p^2 = \frac{\xi_p}{\epsilon_p}, \quad (8.36)$$

where, as before, $\xi_p = \epsilon_p^0 + n_0 U_0$. Thus, when an excitation is added keeping the total number of particles fixed, N_0 must be reduced by the corresponding amount. At large momenta the particle number associated with an excitation tends to unity, since then excitations are just free particles, while for small momenta the effective particle number diverges as ms/p .

The depletion of the ground state at zero temperature may be calculated by evaluating the second term in Eq. (8.34) explicitly and one finds for the number of particles per unit volume in excited states³

$$n_{\text{ex}} = \frac{1}{V} \sum_{\mathbf{p}(\mathbf{p} \neq 0)} v_p^2 = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} v_p^2 = \frac{1}{3\pi^2} \left(\frac{ms}{\hbar} \right)^3, \quad (8.37)$$

which is of order one particle per volume ξ^3 , where ξ is the coherence length, Eq. (6.62). Physically this result may be understood by noting that v_p^2 is of order unity for momenta $p \sim \hbar/\xi$, and then falls off rapidly at larger momenta. The number density of particles in excited states is thus of order the number of states per unit volume with wave number less than $1/\xi$, that is, $1/\xi^3$ in three dimensions. The depletion may also be expressed in terms of the scattering length by utilizing the result (8.30) with $U_0 = 4\pi\hbar^2 a/m$, and one finds

$$\frac{n_{\text{ex}}}{n} = \frac{8}{3\sqrt{\pi}} (na^3)^{1/2}. \quad (8.38)$$

In deriving this result we have assumed that the depletion of the condensate

³ When transforming sums to integrals we shall use the standard prescription $\sum_{\mathbf{p}} \dots = V \int d\mathbf{p}/(2\pi\hbar)^3 \dots$

is small, and (8.38) is therefore only valid when the particle spacing is large compared with the scattering length, or $n_{\text{ex}} \ll n$. In most experiments that have been carried out, the ground-state depletion is of the order of one per cent. Recent experiments on ^{85}Rb near a Feshbach resonance achieved very large values of the scattering length corresponding to a depletion of 10%, thus opening up the possibility of measuring effects beyond the validity of the mean-field approximation [2].

Ground-state energy

The calculation of higher-order contributions to the energy requires that one go beyond the simple approximation in which the effective interaction is replaced by $U_0 = 4\pi\hbar^2 a/m$. The difficulty with the latter approach is seen by considering the expression for the ground-state energy E_0 that one obtains from Eq. (8.28),

$$E_0 = \frac{N^2 U_0}{2V} - \frac{1}{2} \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}}^0 + n_0 U_0 - \epsilon_{\mathbf{p}}). \quad (\text{wrong!}) \quad (8.39)$$

Formally the sum is of order U_0^2 , as one can see by expanding the summand for large p . However, the sum diverges linearly at large p : the leading terms in the summand are of order $1/p^2$, and the sum over momentum space when converted to an integral gives a factor $p^2 dp$. This difficulty is due to the fact that we have used the effective interaction U_0 , which is valid only for small momenta, to calculate high-momentum processes. In perturbation theory language, the effective interaction takes into account transitions to intermediate states in which the two interacting particles have arbitrarily high momenta. If the sum in Eq. (8.39) is taken over all states, contributions from these high-energy intermediate states are included twice. To make a consistent calculation of the ground-state energy one must use an effective interaction $U(p_c)$ in which all intermediate states with momenta in excess of some cut-off value p_c are taken into account, and then evaluate the energy omitting in the sum in the analogue of Eq. (8.39) all intermediate states with momenta in excess of this cut off. The ground-state energy is therefore

$$E_0 = \frac{N^2 U(p_c)}{2V} - \frac{1}{2} \sum_{\mathbf{p}(p < p_c)} (\epsilon_{\mathbf{p}}^0 + n_0 U_0 - \epsilon_{\mathbf{p}}). \quad (8.40)$$

The effective interaction $\tilde{U} = U(p_c)$ for zero energy E and for small values of p_c may be obtained from (5.40) by replacing T by U_0 , which is the effective interaction for $p_c = 0$ and $E = 0$. The imaginary part of the effective interaction, which is due to the $i\delta$ term in the energy denominator in (5.40),

is proportional to $E^{1/2}$, and therefore it vanishes at zero energy. The effective interaction for small p_c and zero energy is thus given by

$$U(p_c) = U_0 + \frac{U_0^2}{V} \sum_{\mathbf{p}(p < p_c)} \frac{1}{2\epsilon_p^0}. \quad (8.41)$$

With this expression for the effective interaction $U(p_c)$ one finds

$$E_0 = \frac{N^2 U_0}{2V} - \frac{1}{2} \sum_{\mathbf{p}(p < p_c)} \left[\epsilon_p^0 + n_0 U_0 - \epsilon_p - \frac{(nU_0)^2}{2\epsilon_p^0} \right]. \quad (8.42)$$

If one chooses the cut-off momentum to be large compared with ms but small compared with \hbar/a the result does not depend on p_c , and, using the fact that $n_0 \simeq n$, one finds

$$\begin{aligned} \frac{E_0}{V} &= \frac{n^2 U_0}{2} + \frac{8}{15\pi^2} \left(\frac{ms}{\hbar} \right)^3 ms^2 \\ &= \frac{n^2 U_0}{2} \left[1 + \frac{128}{15\pi^{1/2}} (na^3)^{1/2} \right]. \end{aligned} \quad (8.43)$$

The first form of the correction term indicates that the order of magnitude of the energy change is the number of states having wave numbers less than the inverse coherence length, times the typical energy of an excitation with this wave number, as one would expect from the form of the integral. This result was first obtained by Lee and Yang [3].

States with definite particle number

The original microscopic Hamiltonian (8.3) conserves the total number of particles. The assumption that the annihilation operator for a particle has a non-zero expectation value, as indicated in Eq. (8.2), implies that the states we are working with are not eigenstates of the particle number operator. In an isolated cloud of gas, the number of particles is fixed, and therefore the expectation value of the particle annihilation operator vanishes. Assuming that the annihilation operator for a particle has a non-zero expectation value is analogous to assuming that the operator for the electromagnetic field due to photons may be treated classically. In both cases one works with coherent states, which are superpositions of states with different numbers of particles or photons. It is possible to calculate the properties of a Bose gas containing a definite particle number by introducing the operators [4]

$$\bar{a}_{\mathbf{p}} = a_0^\dagger (\hat{N}_0 + 1)^{-1/2} a_{\mathbf{p}}, \quad \bar{a}_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger (\hat{N}_0 + 1)^{-1/2} a_0, \quad (\mathbf{p} \neq 0), \quad (8.44)$$

where $\hat{N}_0 = a_0^\dagger a_0$ is the operator for the number of particles in the zero-momentum state. By evaluating the commutators explicitly, one can show

that these operators obey Bose commutation relations when they act on any state which has a non-vanishing number of particles in the zero-momentum state. In the presence of a Bose-Einstein condensate, components of the many-particle state having no particles in the zero-momentum state play essentially no role, so we shall not consider this restriction further. In addition, the operator $\tilde{a}_p^\dagger \tilde{a}_p$ is identically equal to $a_p^\dagger a_p$ for $p \neq 0$. Retaining only terms no more than quadratic in the operators \tilde{a}_p and \tilde{a}_p^\dagger , one may write the Hamiltonian for states that deviate little from the fully-condensed state containing a definite number of particles N as

$$H = \frac{N(N-1)U_0}{2V} + \sum_{p(p \neq 0)}' \left\{ (\epsilon_p^0 + \frac{\hat{N}_0}{V}U_0)(\tilde{a}_p^\dagger \tilde{a}_p + \tilde{a}_{-p}^\dagger \tilde{a}_{-p}) + \frac{U_0}{V} [(\hat{N}_0 + 2)^{1/2}(\hat{N}_0 + 1)^{1/2} \tilde{a}_p^\dagger \tilde{a}_{-p}^\dagger + \tilde{a}_p \tilde{a}_{-p} (\hat{N}_0 + 2)^{1/2}(\hat{N}_0 + 1)^{1/2}] \right\}. \quad (8.45)$$

When one replaces \hat{N}_0 by its expectation value N_0 and neglects terms of relative order $1/N_0$ and $1/N$, this equation becomes identical with Eq. (8.12) apart from the replacement of a_p and a_p^\dagger by \tilde{a}_p and \tilde{a}_p^\dagger . In terms of new operators defined by

$$\tilde{\alpha}_p^\dagger = u_p \tilde{a}_p^\dagger + v_p \tilde{a}_{-p} = u_p a_p^\dagger (\hat{N}_0 + 1)^{-1/2} a_0 + v_p a_0^\dagger (\hat{N}_0 + 1)^{-1/2} a_{-p}, \quad (8.46)$$

which is analogous to Eq. (8.31), the Hamiltonian reduces to Eq. (8.28), but with the operators $\tilde{\alpha}_p$ instead of α_p . This shows that the addition of an elementary excitation of momentum p is the superposition of the addition of a particle of momentum p together with the removal of a particle from the condensate, and the removal of a particle with momentum $-p$ accompanied by the addition of a particle to the condensate. The fact that the total number of particles remains unchanged is brought out explicitly. The physical character of long-wavelength excitations may be seen by using the fact that $u_p \simeq v_p$ in this limit. Therefore, for large N_0 , $\tilde{\alpha}_p^\dagger$ is proportional to $a_p^\dagger a_0 + a_0^\dagger a_{-p}$, which is the condensate contribution to the operator $\sum_{p'} a_{p+p'}^\dagger a_{p'}$ that creates a density fluctuation. This confirms the phonon nature of long-wavelength excitations.

8.2 Excitations in a trapped gas

In Chapter 7 we calculated properties of excitations using a classical approach. The analogous quantum-mechanical theory can be developed along similar lines. It parallels the treatment for the uniform system given in