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## Classical Amplitude Squeezing for Precision Measurements

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In analogy to squeezing of light, noise in a classical oscillator can be squeezed to reduce amplitude uncertainty. While this can be achieved to some extent in a harmonic oscillator parametrically driven at  $2\omega_0$ , true amplitude squeezing is possible in anharmonic oscillators, either by driving at  $2\omega_0$  or allowing amplitude-dependent dephasing. These techniques can reduce the uncertainty in measurements of the frequency of an oscillator; for example, the thermal uncertainty in the relativistic frequency shift in single ion mass spectroscopy can be reduced by more than a factor of 5.

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In recent years, understanding of squeezed light [1,2] has evolved to the point that detection below the shot-noise limit has been demonstrated [3,4], and several applications of these nonclassical states are being considered [5]. Although the emphasis has been in the quantum regime, where the source of noise is the uncertainty principle, there is a classical correspondence [6] which suggests that noise of a thermal or technical origin can be squeezed to minimize its unwanted effects on a particular measurement. Such a reduction of thermal noise in a quadrature component has been observed in a high- $Q$  classical oscillator by parametric excitation and has applications to atomic force microscopy and gravity wave detection [7]. A similar reduction in amplitude uncertainty would be useful for determining the frequency of an anharmonic oscillator in the presence of noise. Since the frequency is amplitude dependent, fluctuations in amplitude will result in fluctuations in the measured frequency. This paper describes three schemes for amplitude squeezing in a classical anharmonic oscillator: by driving parametrically at  $2\omega_0$  in the anharmonic and harmonic regimes, and by dephasing in an undriven oscillator. As an illustration, this concept is applied to high-precision mass spectroscopy of a single trapped ion.

We begin by considering the motion of a classical anharmonic oscillator parametrically driven at twice the resonant frequency. A simple treatment is presented, with emphasis on the phase diagram, of an undamped resonant oscillator to lowest order in the parametric drive

strength and the anharmonicity. (Higher-order expansions, detuning, and damping [8,9] can be neglected for the mass spectroscopy example.) Afterwards, the special cases of no anharmonicity and no parametric drive are considered.

The potential for a one-dimensional oscillator with a small ( $az^2 \ll 1$ ) quartic anharmonic correction whose frequency is modulated at  $2\omega_0$  by a weak ( $\varepsilon \ll 1$ ) parametric drive is

$$U(z, t) = \frac{1}{2} m \omega_0^2 z^2 (1 + \varepsilon \sin 2\omega_0 t + \frac{1}{2} a z^2). \quad (1)$$

To lowest order in  $a$  and  $\varepsilon$ , higher harmonics can be neglected, and one expects oscillation only at  $\omega_0$ :

$$\begin{aligned} z(t) &= r(t) \cos(\omega_0 t - \theta(t)) \\ &= C(t) \cos \omega_0 t + S(t) \sin \omega_0 t. \end{aligned} \quad (2)$$

In this approximation,  $C(t)$  and  $S(t)$  are slowly varying (i.e.,  $dC/dt, dS/dt \ll r\omega_0$ ). Thus  $d^2C/dt^2$  and  $d^2S/dt^2$  can be neglected in the equation of motion, yielding the autonomous system of equations

$$\frac{dC}{dt} = \kappa(C + \gamma r^2 S), \quad \frac{dS}{dt} = -\kappa(S + \gamma r^2 C), \quad (3)$$

where

$$\kappa = \varepsilon \omega_0 / 4, \quad \gamma = 3a / 2\varepsilon.$$

Rewriting in terms of  $r$  and  $\theta$  leads to the first integral of the motion:

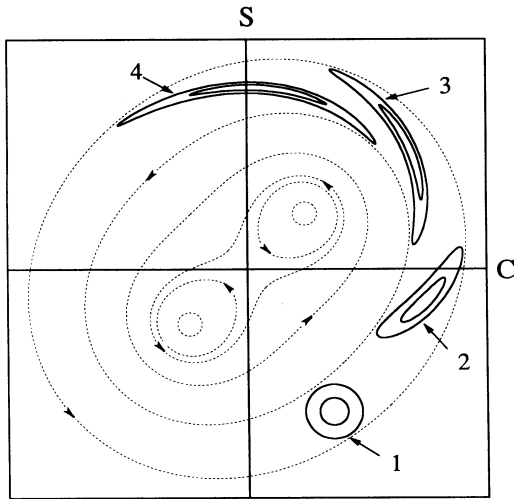


FIG. 1. Amplitude squeezing for an anharmonic oscillator driven at  $2\omega_0$  for  $\gamma < 0$ . Initial Gaussian contours (1) are transformed along the phase trajectories in the  $C$ - $S$  plane (dotted lines) through intermediate states (2) and (3) until amplitude uncertainty is reduced (4). (The  $\gamma > 0$  diagram is obtained by reflecting about either axis.)

$$a^4 = r_1^2 r_2^2 = r^4 + 2r^2 \sin 2\theta / \gamma + 1/\gamma^2, \quad (4)$$

which specifies the trajectories in phase space (Fig. 1). They are Cassinian ovals, defined as the loci of points which maintain a constant product of the distances  $r_1$  and  $r_2$  from the two fixed points (centers) located at  $(r^*, \theta^*)$ :

$$r^* = |\gamma|^{-1/2}, \quad \theta^* = \text{sgn}(\gamma) \frac{1}{4} \pi \pm \frac{1}{2} \pi. \quad (5)$$

This enables the system of equations to be decoupled, and the time evolution along the trajectories is obtained by solving the resultant elliptic integral equation for  $r(t)$  or  $\theta(t)$ . There are two topologically distinct types of phase trajectories. The "outer" trajectories (case  $a > r^*$ ) encircle both centers, and the time evolution is determined from

$$\kappa t = - (1/2\gamma a^2) [F(\Theta_>(\theta) | 1/\gamma^2 a^4)]_{\theta=\theta(0)}^{\theta=\theta(t)}, \quad (6)$$

where

$$\sin \Theta_>(\theta) = \frac{\sin 2\theta}{(1 - \cos^2 2\theta / \gamma^2 a^4)^{1/2}}$$

and  $F(\phi|m)$  is the Legendre elliptic integral of the first kind [10]. The "inner" trajectories (case  $0 < a < r^*$ ) encircle just one of the centers. From (4), there are two solutions for  $r$ , and the trajectories evolve according to

$$\kappa t = \frac{1}{2} [\pm F(\Theta_<(\theta) | \gamma^2 a^4)]_{\theta=\theta(0)}^{\theta=\theta(t)}, \quad (7)$$

where

$$\sin \Theta_<(\theta) = \cos 2\theta / |\gamma a^2|.$$

This phase diagram can be qualitatively understood as

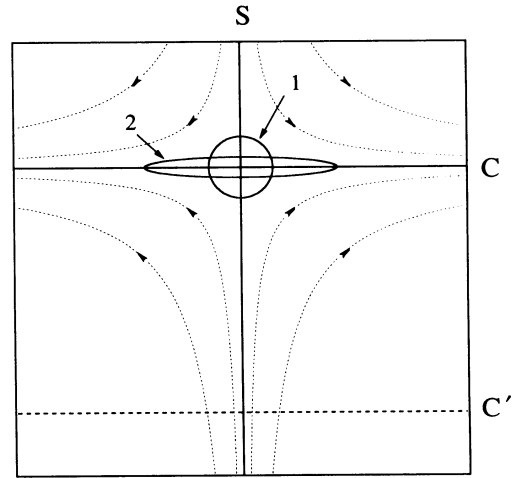


FIG. 2. Quadrature squeezing for a harmonic oscillator driven at  $2\omega_0$ . An initial Gaussian distribution (1) at the origin is transformed along the hyperbolic phase trajectories into an elliptical distribution (2). A coherent pulse displaces the distribution so that its major axis is in the tangential direction, resulting in a reduced amplitude uncertainty with respect to the final coordinates ( $C'$ ,  $S$ ).

a competition between the parametric excitation and the anharmonicity. The parametric drive amplifies the in-phase component and attenuates the out-of-phase component, resulting in a flow towards large  $|C|$  and small  $|S|$ . The anharmonicity causes dephasing, appearing as a rotation about the origin. At the centers, the two effects cancel.

The well-known cases of the parametrically driven harmonic oscillator and the undriven anharmonic oscillator can be recovered by letting  $a \rightarrow 0$  and  $\epsilon \rightarrow 0$ , respectively. For the driven harmonic case, the only fixed point is a saddle point at the origin. The phase trajectories are hyperbolic (Fig. 2), flowing towards large  $|C|$  and small  $|S|$ . They are specified by the constant of the motion  $\lambda' = r^2 \sin 2\theta$ , and their time evolution is given by  $C(t) = C(0)e^{\kappa t}$  and  $S(t) = S(0)e^{-\kappa t}$ . For the undriven anharmonic case, the fixed points merge into a center at the origin. The amplitude  $r$  is the constant of the motion, and the phase trajectories are concentric circles (Fig. 3) evolving according to  $\theta(t) = \kappa \gamma r^2 t / \omega_0$ .

The above equations can be used to study how the thermal noise statistics are affected by parametric amplification and anharmonicity. The noise is simply modeled so that the quadrature components  $C$  and  $S$  for an ensemble of identically cooled oscillators are Gaussian with equal standard deviations:  $\Delta C = \Delta S = \sigma$ . If this noise is displaced to a large mean amplitude ( $\langle r \rangle \gg \sigma$ ), the polar components  $r$  and  $\theta$  are approximately Gaussian near the center of the distribution. In this limit, the amplitude uncertainty and the phase uncertainty are about equal:  $\Delta r \cong \langle r \rangle \Delta \theta \cong \sigma$ .

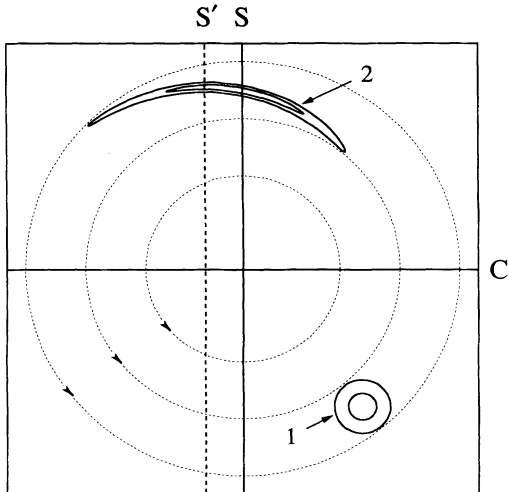


FIG. 3. Amplitude squeezing for an undriven anharmonic oscillator. An initial Gaussian distribution (1) dephases with constant amplitude, becoming a crescent distribution (2). After a C displacement in the tangential direction, the distribution is amplitude squeezed with respect to the new coordinates (C, S').

The thermal noise can be “amplitude squeezed” by a transformation which produces a crescent-shaped distribution while maintaining the same phase-space density (by Liouville’s theorem), thereby reducing the amplitude uncertainty at the expense of the phase uncertainty. If this process is fast enough, further thermal effects may be ignored. An ideal amplitude-squeezing process preserves the product  $\langle r \rangle \Delta r \Delta \theta$ , just as a minimum uncertainty quantum state. This fact can be used to characterize the quality of squeezing by defining a “squeezing efficiency”  $\eta$  as

$$\eta = \sigma^2 / \langle r \rangle \Delta r \Delta \theta, \tag{8}$$

which is unity for a “minimum uncertainty” classical distribution.

In the driven anharmonic oscillator, the initial circular distribution is distorted into a crescent along the outer trajectories, as shown in Fig. 1. Amplitude squeezing occurs where the flow is toward smaller amplitude, in this case near the S axis. The points with larger amplitude move faster in phase space and reach this region first, thereby reducing the amplitude uncertainty. Squeezing efficiencies close to 1 can be attained by this method. Reduced amplitude uncertainty can also result from propagation along the inner trajectories, but to a much lesser extent.

Quadrature squeezing can be achieved by driving the oscillator at  $2\omega_0$  in the harmonic regime. The initial circular distribution becomes elliptical, preserving the relationship  $\Delta C \Delta S = \sigma^2$ . In order to achieve reduction in  $\Delta r$ , a coherent force pulse must be applied to displace the distribution such that the major axis is along the tangential direction [5] (Fig. 2). Although there is no limit to the

amount of quadrature squeezing in a harmonic oscillator, there is a limit for amplitude squeezing since the final distribution is straight rather than crescent shaped. The tails of the distribution are at a larger amplitude; thus only a limited reduction in  $\Delta r$  is possible.

A third method for amplitude squeezing is by the dephasing of an undriven anharmonic oscillator, which is analogous to the production of amplitude-squeezed light by self-phase modulation [11]. The outermost points dephase faster, changing the initial circle into a crescent, as can be seen in Fig. 3. Since the amplitude is constant in the absence of the parametric drive, the amplitude uncertainty cannot be reduced by this process alone. It is necessary to displace the distribution in the tangential direction by applying a coherent pulse. The result is an amplitude-squeezed distribution with an efficiency close to 1.

The techniques discussed so far can have important applications in single-ion mass spectroscopy. To date, relative mass measurements have been made up to an accuracy of  $4 \times 10^{-10}$  in a Penning trap [12,13]. When the precision is improved to the  $10^{-11}$  range, relativistic effects will cause significant error if the mode amplitudes are known only to their thermal cooling limits. To lowest order, the velocity of a trapped ion is  $v = \omega'_c \rho_c$ , where  $\omega'_c \cong eB/mc$  and  $\rho_c$  are the trap cyclotron frequency and radius. Thus the relativistic mass shift  $\delta m$  is amplitude dependent, and the thermal uncertainty in  $\rho_c$  will cause a mass uncertainty of

$$\Delta \left( \frac{\delta m}{m} \right) = \frac{\omega_c'^2}{2c^2} \Delta(\rho_c^2) \cong \frac{\omega_c'^2}{c^2} \langle \rho_c \rangle \Delta \rho_c. \tag{9}$$

A single trapped ion [14] is well suited for the above squeezing schemes. The ion’s amplitude and phase can be controlled by applying coherent pulses to the endcaps, and the trap anharmonicity can be adjusted by tuning the compensation electrodes. The ion can be decoupled from the resonant detector by changing the trapping potential, becoming effectively undamped. By squeezing the thermal noise, the precision can be improved by the factor  $\beta$ :

$$\beta = \frac{\Delta(\rho_c^2)(\text{unsqueezed})}{\Delta(\rho_c^2)(\text{squeezed})}. \tag{10}$$

Computer simulations of the evolution of an initial Gaussian noise distribution of 10000 points were done using typical experimental parameters to calculate the squeezing efficiencies and error reductions that would be possible. The final amplitude was adjusted to be 20 times the initial rms amplitude after cooling, ensuring that the ion would stay within the region where the anharmonicity can be controlled. All the additional parameters were adjusted to maximize the error reduction  $\beta$  for a specific squeezing time. Table I summarizes  $\beta$  and  $\eta$  for the three methods. Both anharmonic schemes can achieve true amplitude squeezing ( $\eta \cong 1$ ) with large error reduc-

TABLE I. Maximum error reduction  $\beta$  and typical squeezing efficiency  $\eta$  (at  $\beta=5$ ) for single-ion mass spectroscopy simulations where  $\langle r \rangle = 20\sqrt{2}\sigma$ .

	$\beta(\text{max})$	$\eta$ (for $\beta=5$ )
Driven anharmonic	5.8	0.97
Driven harmonic	2.5	...
Undriven anharmonic	8.7	0.98

tions ( $\beta > 5$ ), which cannot be achieved by the harmonic scheme. The limit in  $\beta$  occurs when the tails of the distribution do not follow the proper curvature.

Each of the classical squeezing methods which have been discussed has its advantages and disadvantages for application to this experiment. The harmonic squeeze is easiest to implement because there is no need to introduce anharmonicity, but it has a low  $\beta$  and  $\eta$ . Nonetheless, because of its simplicity, the harmonic squeeze would be useful as a first step before exploring the anharmonic methods. The main advantage of the driven anharmonic squeeze is that a final pulse is not needed. The undriven anharmonic squeeze offers the most error reduction, but like the harmonic squeeze, it requires a final pulse whose phase must be precisely controlled with respect to the evolved distribution. In this experiment, however, having  $\beta$  greater than about 5 is not desirable because the increased phase uncertainty substantially increases the measurement time. Also, since the same measurement is performed many times, a squeezing process with fewer steps would be more practical. Thus the driven anharmonic method appears to be the most appropriate for this application.

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