Photon-Atom Interactions

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Then

$$(\Delta Q)_c^2 \equiv \langle\alpha|Q^2|\alpha\rangle - \langle\alpha|Q|\alpha\rangle^2 = \frac{\hbar}{2\omega},$$

$$(\Delta P)_c^2 \equiv \langle\alpha|P^2|\alpha\rangle - \langle\alpha|P|\alpha\rangle^2 = \frac{\hbar\omega}{2}, \tag{4.271}$$

$$(\Delta Q\,\Delta P)_c = \frac{\hbar}{2}. \tag{4.272}$$

The coherent state is therefore a *minimum uncertainty state*—a feature shared with the vacuum state in the photon-number basis.

4.9 Squeezed States

Squeezed states are among the discoveries involving laser radiation that have an important bearing on potential applications to low-noise detection systems. We begin by writing the annihilation and creation operators for a single mode in the form

$$a = \frac{1}{\sqrt{2}}(a_1 + ia_2), \qquad a^\dagger = \frac{1}{\sqrt{2}}(a_1 - ia_2),$$
$$a_1 = \frac{1}{\sqrt{2}}(a^\dagger + a), \qquad a_2 = \frac{i}{\sqrt{2}}(a^\dagger - a), \tag{4.273}$$

in which a_1 and a_2, known as the *quadrature components* of the radiation mode, are Hermitian operators. They are related closely to the position and momentum operators of the field since, according to Eq. (4.95),

$$Q = \sqrt{\frac{\hbar}{2\omega}}(a^\dagger + a) = \sqrt{\frac{\hbar}{\omega}}\,a_1,$$
$$P = i\sqrt{\frac{\hbar\omega}{2}}(a^\dagger - a) = \sqrt{\hbar\omega}\,a_2. \tag{4.274}$$

The reason for referring to a_1 and a_2 as quadrature components may be seen by writing the electric field operator in terms of a_1 and a_2. For a single mode in the Heisenberg representation, we have, from Eq. (4.119),

$$\mathbf{E}_H(\mathbf{r},t) = K\hat{\varepsilon}i[ae^{i\theta} - a^\dagger e^{-i\theta}]$$
$$= \frac{K}{\sqrt{2}}\hat{\varepsilon}i[(a_1 + ia_2)e^{i\theta} - (a_1 - ia_2)e^{-i\theta}]$$
$$= -K\hat{\varepsilon}\sqrt{2}(a_2\cos\theta + a_1\sin\theta) \tag{4.275}$$

in which

$$K = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}, \qquad \theta = \mathbf{k}\cdot\mathbf{r} - \omega t. \tag{4.276}$$

Thus, in terms of a_1 and a_2 (or in terms of P and Q), the two components of the radiation mode are 90° out of phase (i.e., in quadrature).

The commutator relation $[a, a^\dagger] = 1$ enables us to derive a commutator relation for a_1 and a_2. Applying the definitions (4.273), we find

$$[a, a^\dagger] = i[a_2, a_1] = 1 \tag{4.277a}$$

or

$$[a_1, a_2] = i. \tag{4.277b}$$

But the general uncertainty principle states that if two observables A and B satisfy the commutator relation $[A, B] = ic$ where c is a numerical constant, then $\Delta A\,\Delta B \geq c/2$. Therefore, in view of Eq. (4.277), we have, in the photon-number basis,

$$\Delta a_1\,\Delta a_2 = \frac{1}{2}(2n+1) \geq \frac{1}{2}. \tag{4.278}$$

The same result may be deduced from the position-momentum uncertainty relation $\Delta Q\,\Delta P \geq \hbar/2$.

The variances of Q and P in a coherent state basis were given given by Eq. (4.271); thus,

$$(\Delta Q)_c^2 = \frac{\hbar}{\omega}(\Delta a_1)_c^2 = \frac{\hbar}{2\omega}, \qquad (\Delta P)_c^2 = \hbar\omega(\Delta a_2)_c^2 = \frac{\hbar\omega}{2}. \tag{4.279}$$

Then

$$(\Delta a_1)_c^2 = (\Delta a_2)_c^2 = \frac{1}{2}, \qquad (\Delta a_1\,\Delta a_2)_c = \frac{1}{2}, \tag{4.280}$$

which also may be obtained directly from Eq. (4.273). We conclude that in a coherent state, which is a minimun uncertainty state as shown by Eq. (4.272), the variances of the two quadrature components are equal. The same property is found for the vacuum state in the photon-number basis, as is evident from Eq. (4.269). *Squeezed states* [9–11] are members of a broader class of minimum uncertainty states in which the variances of the two quadrature components are *not* equal. This implies that the quantum fluctuations in one quadrature component may be reduced at the expense of increased quantum fluctuations in the other quadrature component. Or, in terms of an electromagnetic field consisting of two components that are 90° out of phase, as in Eq. (4.275), squeezing means that the quantum noise in one component is

reduced while the other component suffers an increase in quantum noise. The condition for squeezing therefore, may, be written

$$\Delta a_1\,\Delta a_2 = \frac{1}{2}, \qquad \text{but } (\Delta a_i)^2 < \frac{1}{2} \text{ for } i = 1 \text{ or } 2. \tag{4.281}$$

We note that a photon-number state is not a minimum-uncertainty state, as shown by Eq. (4.269), and therefore cannot be squeezed.

Squeezed states may be generated by means of a *squeeze operator* defined by

$$S(z) = \exp\left[\frac{1}{2}(za^2 - z^* a^{\dagger 2})\right] \equiv e^{iA}, \qquad z = re^{-i\phi}. \tag{4.282}$$

Since the operator

$$A = -\frac{i}{2}(za^2 - z^* a^{\dagger 2}) \tag{4.283}$$

is Hermitian, $S(z)$ must be unitary, i.e.,

$$S^\dagger(z) = S^{-1}(z). \tag{4.284}$$

Let us now perform a unitary transformation on the annihilation operator a by means of the squeeze operator:

$$\begin{aligned} t &\equiv S(z)aS^\dagger(z) = e^{iA}ae^{-iA} \\ &= a + [iA, a] + \frac{1}{2!}[iA, [iA, a]] + \frac{1}{3!}[iA, [iA, a]] + \cdots \end{aligned} \tag{4.285}$$

in which the theorem (4.170) has been used. With the commutation property $[a, a^\dagger] = 1$, the result is

$$\begin{aligned} t &= a + z^* a^\dagger + \frac{1}{2!}|z|^2 a + \frac{1}{3!}|z|^2 z^* a^\dagger + \frac{1}{4!}|z|^4 a + \cdots, \\ &= a\left(1 + \frac{1}{2!}r^2 + \frac{1}{4!}r^4 + \cdots\right) + a^\dagger e^{i\phi}\left(r + \frac{1}{3!}r^3 + \frac{1}{5!}r^5 \cdots\right), \end{aligned} \tag{4.286a}$$

or

$$t = a\cos hr + a^\dagger e^{i\phi}\sin hr, \qquad t^\dagger = a^\dagger \cos hr + ae^{-i\phi}\sin hr, \tag{4.286b}$$

which also is known as the *Bogoliubov transformation*. The transformed operators t and $t^\dagger$ then satisfy

$$t = \mu^* a - \nu a^\dagger, \qquad |\mu|^2 - |\nu|^2 = 1, \tag{4.287}$$

and the commutation rule

$$[t, t^\dagger] = 1. \tag{4.288}$$

We now define the c-numbers τ and τ^*:

$$\tau = \alpha \cos hr + \alpha^* e^{i\phi} \sin hr, \qquad \tau^* = \alpha^* \cos hr + \alpha e^{-i\phi} \sin hr, \quad (4.289a)$$

$$\tau t^\dagger - \tau^* t = \alpha a^\dagger - \alpha^* a. \quad (4.289b)$$

The definition of a squeezed state follows a pattern similar to the definition of a coherent state, which was shown to satisfy the relations

$$a|\alpha\rangle = \alpha|\alpha\rangle, \qquad a|O\rangle = O,$$
$$|\alpha\rangle = D(\alpha)|O\rangle, \qquad D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}. \quad (4.290)$$

We now may define an operator,

$$D(\tau) = e^{\tau t^\dagger - \tau^* t} = e^{\alpha a^\dagger - \alpha^* a} = D(\alpha), \quad (4.291)$$

in which the equality $D(\tau) = D(\alpha)$ follows from Eq. (4.286b). Whereas the coherent state $|\alpha\rangle$ is generated from the vacuum state $|O\rangle$ by the displacement operator $D(\alpha)$, the squeezed state $|\tau\rangle$ is generated from the vacuum by the relation

$$|\tau\rangle = D(\alpha)S(z)|O\rangle. \quad (4.292)$$

The vacuum state with respect to the transformed annihilation operator t, written $|O_t\rangle$, is defined by

$$|O_t\rangle = S(z)|O\rangle, \quad (4.293)$$

since

$$t|O_t\rangle = tS(z)|O\rangle = S(z)aS^\dagger(z)S(z)|O\rangle$$
$$= S(z)a|O\rangle = O. \quad (4.294)$$

Substituting Eqs. (4.293) and (4.291) into Eq. (4.292), it is seen that

$$|\tau\rangle = D(\tau)|O_t\rangle, \qquad D^{-1}(\tau)|\tau\rangle = |O_t\rangle; \quad (4.295)$$

that is, the squeezed state is generated by the displacement operator $D(\tau)$ from the vacuum state with respect to the operator t. This is an exact parallel to the manner in which a coherent state is generated. Furthermore, it may be shown that $|\tau\rangle$ is an eigenstate of t with the complex eigenvalue τ. Thus, noting that $D(\tau)$ is subject to the same type of analysis as that leading to Eq. (4.179), we have

$$D^{-1}(\tau)tD(\tau) = t + \tau, \qquad D^{-1}(\tau)t^\dagger D(\tau) = t^\dagger + \tau^*, \quad (4.296a)$$

or

$$t = D(\tau)(t + \tau)D^{-1}(\tau), \qquad t^\dagger = D(\tau)(t^\dagger + \tau^*)D^{-1}(\tau). \quad (4.296b)$$

Therefore, in view of Eqs. (4.296b) and (4.294),

$$t|\tau\rangle = D(\tau)(t + \tau)D^{-1}(\tau)|\tau\rangle = D(\tau)tD^{-1}(\tau)|\tau\rangle + \tau|\tau\rangle$$
$$= D(\tau)t|O_t\rangle + \tau|\tau\rangle = \tau|\tau\rangle. \quad (4.297)$$

One may regard Eq. (4.297) as an alternative definition of a squeezed state.

It may appear that up to this point nothing has been accomplished other than to duplicate the formalism of the coherent state with a change of notation. Nevertheless, it will be shown shortly that the statistics of the squeezed states differ in an important respect from the statistics of coherent states. The commutation rule (Eq. (4.288)) and the definition (4.297) for the squeezed state enable us to write expressions similar in form to those in Eq. (4.143) as, for example

$$\langle\tau|(t^\dagger)^m t^n|\tau\rangle = (\tau^*)^m \tau^n, \qquad \langle\tau|tt^\dagger|\tau\rangle = |\tau|^2 + 1. \quad (4.298)$$

We now shall transform the quadrature components a_1 and a_2 with $\phi = 0$:

$$t_1 = \frac{1}{\sqrt{2}}(t^\dagger + t) = \frac{1}{\sqrt{2}}S(z)(a^\dagger + a)S^\dagger(z)$$
$$= \frac{1}{\sqrt{2}}(a^\dagger + a)(\cos hr + \sin hr) = a_1 e^r, \quad (4.299a)$$

$$t_2 = \frac{i}{\sqrt{2}}(t^\dagger - t) = \frac{i}{\sqrt{2}}S(z)(a^\dagger - a)S^\dagger(z)$$
$$= \frac{i}{\sqrt{2}}(a^\dagger - a)(\cos hr - \sin hr) = a_2 e^{-r}. \quad (4.299b)$$

These relations, together with Eq. (4.298), make it possible to demonstrate the essential difference between a coherent state and a squeezed state: coherent and squeezed states share the common property of minimum uncertainty but whereas the variances of a_1 and a_2 are equal in a coherent state, they are not equal in a squeezed state. For this purpose, it is necessary to compute several matrix elements. With the understanding that $\phi = 0$,

$$\langle\tau|t_1|\tau\rangle = \frac{1}{\sqrt{2}}\langle\tau|t^\dagger + t|\tau\rangle = \frac{1}{\sqrt{2}}(\tau^* + \tau) \quad (4.300)$$

$$\langle\tau|t_1^2|\tau\rangle = \frac{1}{2}\langle\tau|t^{\dagger 2} + t^\dagger t + tt^\dagger + t^2|\tau\rangle = \frac{1}{2}(\tau^{*2} + 2|\tau|^2 + \tau^2 + 1), \quad (4.301)$$

$$(\Delta t_1)^2 \equiv \langle\tau|t_1^2|\tau\rangle - \langle\tau|t_1|\tau\rangle^2 = \frac{1}{2}. \quad (4.302)$$

In the same way,

$$(\Delta t_2)^2 = \frac{1}{2}. \tag{4.303}$$

From Eq. (4.299), however,

$$\langle\tau|t_1|\tau\rangle = \langle\tau|a_1|\tau\rangle e^{r}, \qquad \langle\tau|t_1^2|\tau\rangle = \langle\tau|a_1^2|\tau\rangle e^{2r}, \tag{4.304a}$$

$$\langle\tau|t_2|\tau\rangle = \langle\tau|a_2|\tau\rangle e^{-r}, \qquad \langle\tau|t_2^2|\tau\rangle = \langle\tau|a_2^2|\tau\rangle e^{-2r}. \tag{4.304b}$$

Consequently,

$$(\Delta a_1)_\tau^2 = (\Delta t_1)^2 e^{-2r} = \frac{1}{2}e^{-2r}, \tag{4.305a}$$

$$(\Delta a_2)_\tau^2 = (\Delta t_2)^2 e^{2r} = \frac{1}{2}e^{2r}, \tag{4.305b}$$

$$(\Delta a_1\, \Delta a_2)_\tau = \frac{1}{2}. \tag{4.305c}$$

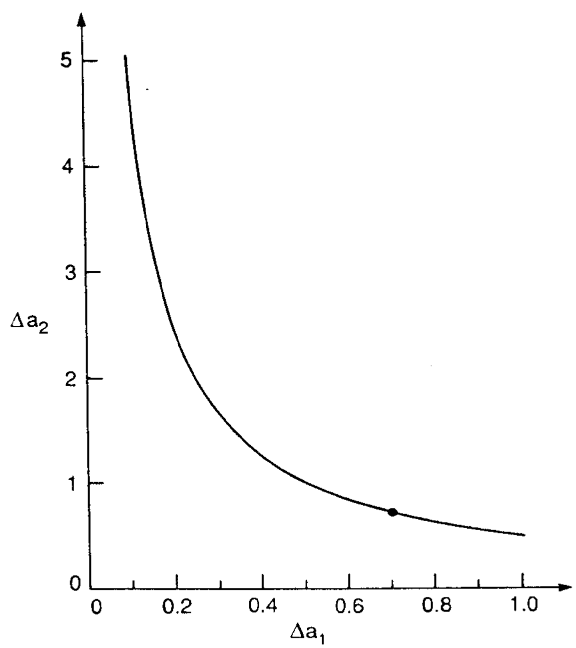

FIGURE 4.3 A plot of Δa_2 *vs* Δa_1 where $\Delta a_1\,\Delta a_2 = 1/2$. The coherent state is the single point at $\Delta a_1 = \Delta a_2 = (1/2)^{1/2}$. Squeezed states correspond to other points on the curve where $\Delta a_1 \neq \Delta a_2$.

The conclusion, then, is that in a squeezed state the magnitude of the fluctuations or the noise level in one of the two quadrature components can be reduced below the noise level of a coherent state, which, we saw, corresponds to the vacuum limit. Clearly, the noise diminution in one quadrature is achieved at the expense of noise enhancement in the other quadrature. A plot of Eq. (4.305) is shown in Fig. 4.3.

Various nonlinear processes have been proposed for the production of squeezed states. Among them are parametric amplifiers [12], degenerate and nondegenerate four-wave mixers [13–17], resonance fluorescence [18, 19], and interaction of electromagnetic waves with plasmas [20]. Experimental verification employing phase-sensitive detection methods has been demonstrated by four-wave mixing [21–23] and by degenerate parametric down-conversion [24]. Applications of squeezed states to communication [25] and to gravitational radiation detectors [26] have been suggested.

4.10 Gauge Transformations

In classical electromagnetic theory, a gauge transformation on the scalar potential $\phi(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$ is defined by the relations

$$\begin{aligned} \phi'(\mathbf{r}, t) &= \phi(\mathbf{r}, t) - \frac{\partial}{\partial t} f(\mathbf{r}, t), \\ \mathbf{A}'(\mathbf{r}, t) &= \mathbf{A}(\mathbf{r}, t) + \nabla f(\mathbf{r}, t), \end{aligned} \tag{4.306}$$

where $f(\mathbf{r}, t)$ is an arbitrary differentiable scalar function of space and time. Since the fields are related to the potentials by

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\nabla\phi(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t), \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t), \end{aligned} \tag{4.307}$$

a gauge transformation of the potentials has no effect on the fields; that is, when ϕ and $\mathbf{A}$ are replaced by ϕ' and $\mathbf{A}'$, the fields are unchanged. Hence, the potentials are not determined uniquely.

It now will be shown that the transformations (4.306) also may arise in a quantum mechanical context. The fundamental Hamiltonian for the interaction of a nonrelativistic electron with an external electromagnetic field [1, 2] is

$$\mathscr{H}(\mathbf{r}, t) = \frac{1}{2m_e}[\mathbf{p} + e\mathbf{A}(\mathbf{r}, t)]^2 - e\phi(\mathbf{r}, t), \tag{4.308}$$