

# Q E D

Maxwell eq.

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{1}{\epsilon_0} \rho(\mathbf{r}, t) \quad (1.a)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (1.b)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \quad (1.c)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t) + \frac{1}{\epsilon_0 c^2} \mathbf{j}(\mathbf{r}, t) \quad (1.d)$$

Spatial Fourier transform

$$i\mathbf{k} \cdot \mathcal{E}(\mathbf{k}, t) = \frac{1}{\epsilon_0} \rho(\mathbf{k}, t) \quad (2.a)$$

$$i\mathbf{k} \cdot \mathcal{B}(\mathbf{k}, t) = 0 \quad (2.b)$$

$$i\mathbf{k} \times \mathcal{E}(\mathbf{k}, t) = -\frac{\partial}{\partial t} \mathcal{B}(\mathbf{k}, t) \quad (2.c)$$

$$i\mathbf{k} \times \mathcal{B}(\mathbf{k}, t) = \frac{1}{c^2} \frac{\partial \mathcal{E}(\mathbf{k}, t)}{\partial t} + \frac{1}{\epsilon_0 c^2} \mathbf{j}(\mathbf{k}, t) \quad (2.d)$$

transverse and long. field decouple

$$\mathbf{E}_{\parallel}(\mathbf{r}, t) = \frac{-1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}', t) \nabla_{\mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad (5.a)$$

$$\mathbf{B}_{\parallel}(\mathbf{r}, t) = 0. \quad (5.b)$$

$$\frac{\partial}{\partial t} \mathcal{B}(\mathbf{k}, t) = -i\mathbf{k} \times \mathcal{E}_{\perp}(\mathbf{k}, t) \quad (6.a)$$

$$\frac{\partial}{\partial t} \mathcal{E}_{\perp}(\mathbf{k}, t) = c^2 i\mathbf{k} \times \mathcal{B}(\mathbf{k}, t) - \frac{1}{\epsilon_0} \mathbf{j}_{\perp}(\mathbf{k}, t). \quad (6.b)$$

# Vector Potential

$$\mathbf{E}(\mathbf{r}, t) = -\nabla U(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad (7.a)$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \quad (7.b)$$

$$\mathcal{E}(\mathbf{k}, t) = -ik\mathcal{U}(\mathbf{k}, t) - \frac{\partial \mathcal{A}(\mathbf{k}, t)}{\partial t} \quad (8.a)$$

$$\mathcal{B}(\mathbf{k}, t) = i\mathbf{k} \times \mathcal{A}(\mathbf{k}, t). \quad (8.b)$$

• Coulomb Gauge  $\nabla \cdot \mathbf{A} = 0$

$$A_{||}(\mathbf{r}, t) = 0$$

$$U(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}. \quad (15)$$

• Normal modes

$$\begin{aligned} \alpha(\mathbf{k}, t) &= \lambda(k) \left[ \mathcal{E}_{\perp}(\mathbf{k}, t) - c \frac{\mathbf{k}}{k} \times \mathcal{B}(\mathbf{k}, t) \right] \\ &= \lambda(k) \left[ -\dot{\mathcal{A}}_{\perp}(\mathbf{k}, t) + i\omega \mathcal{A}_{\perp}(\mathbf{k}, t) \right] \end{aligned} \quad (16)$$

$$\dot{\alpha}(\mathbf{k}, t) + i\omega \alpha(\mathbf{k}, t) = \frac{i}{\sqrt{2\epsilon_0 \hbar \omega}} \dot{\mathcal{A}}_{\perp}(\mathbf{k}, t). \quad (17)$$

$$\vec{A}_{\perp}(\mathbf{r}) = \int d^3k \sum_{\epsilon} A_{\omega} \left[ \vec{\epsilon} \alpha_{\epsilon}(k) e^{i\mathbf{k}\cdot\mathbf{r}} + \vec{\epsilon} \alpha_{\epsilon}^*(k) e^{-i\mathbf{k}\cdot\mathbf{r}} \right]$$

# Field Quantization

$$\pi_{\epsilon}(\mathbf{r}) = \epsilon_0 \dot{A}_{\epsilon}(\mathbf{r}) \quad \text{conjugate momentum}$$

$$[A_{\epsilon}(\mathbf{k}), \pi_{\epsilon'}^+(\mathbf{k}')] = i\hbar \delta_{\epsilon\epsilon'} \delta(\mathbf{k} - \mathbf{k}'). \quad (20.a)$$

$$[a_{\epsilon}(\mathbf{k}), a_{\epsilon'}^+(\mathbf{k}')] = \delta_{\epsilon\epsilon'} \delta(\mathbf{k} - \mathbf{k}') \quad (20.b)$$

## Quantum fields

$$\mathbf{A}_{\perp}(\mathbf{r}) = \int d^3k \sum_{\epsilon} \mathcal{A}_{\omega} [\epsilon a_{\epsilon}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} + \epsilon a_{\epsilon}^+(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (21)$$

$$\mathbf{E}_{\perp}(\mathbf{r}) = \int d^3k \sum_{\epsilon} i\mathcal{E}_{\omega} [\epsilon a_{\epsilon}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} - \epsilon a_{\epsilon}^+(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (22)$$

$$\mathbf{B}(\mathbf{r}) = \int d^3k \sum_{\epsilon} i\mathcal{B}_{\omega} [(\boldsymbol{\kappa} \times \boldsymbol{\epsilon}) a_{\epsilon}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} - (\boldsymbol{\kappa} \times \boldsymbol{\epsilon}) a_{\epsilon}^+(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (23)$$

## Particle Operators

$$[r_i, p_j] = i\hbar \delta_{ij}$$
$$m_{\alpha} \mathbf{v}_{\alpha} = \mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{r}_{\alpha}) \quad (37)$$

$$\rho(\mathbf{r}) = \sum_{\alpha} q_{\alpha} \delta(\mathbf{r} - \mathbf{r}_{\alpha}) \quad (38.a)$$

$$\mathbf{j}(\mathbf{r}) = \sum_{\alpha} q_{\alpha} \mathbf{v}_{\alpha} \delta(\mathbf{r} - \mathbf{r}_{\alpha}). \quad (38.b)$$

# • Hamiltonian

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$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{r}_{\alpha})]^2 + \sum_{\alpha} \left( -g_{\alpha} \frac{q_{\alpha}}{2m_{\alpha}} \right) \mathbf{S}_{\alpha} \cdot \mathbf{B}(\mathbf{r}_{\alpha}) + V_{\text{Coul}} + H_R. \quad (40)$$

$$V_{\text{Coul}} = \frac{\epsilon_0}{2} \int d^3r \mathbf{E}_{\parallel}^2(\mathbf{r}) = \frac{\epsilon_0}{2} \int d^3k |\mathcal{E}_{\parallel}(\mathbf{k})|^2 \quad (41)$$

$$V_{\text{Coul}} = \frac{1}{2\epsilon_0} \int d^3k \frac{\rho^*(\mathbf{k})\rho(\mathbf{k})}{k^2} = \sum_{\alpha} \epsilon_{\text{Coul}}^{\alpha} + \frac{1}{8\pi\epsilon_0} \sum_{\alpha \neq \beta} \frac{q_{\alpha}q_{\beta}}{|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|}. \quad (42)$$

$$\epsilon_{\text{Coul}}^{\alpha} = \frac{q_{\alpha}^2}{2\epsilon_0} \int \frac{d^3k}{(2\pi)^3 k^2} = \frac{q_{\alpha}^2}{4\epsilon_0\pi^2} \int_0^{k_c} dk = \frac{q_{\alpha}^2 k_c}{4\epsilon_0\pi^2}. \quad (43)$$

Coulomb self energy

$$H_R = \frac{\epsilon_0}{2} \int d^3r [\mathbf{E}_{\perp}^2(\mathbf{r}) + c^2 \mathbf{B}^2(\mathbf{r})] \quad (44)$$

$$H_R = \sum_i \hbar\omega_i \left( a_i^{\dagger} a_i + \frac{1}{2} \right). \quad (45)$$

$(N_i + \frac{1}{2})$

# • Splitting the Hamiltonian

$$H = H_P + H_R + H_I \quad (49)$$

$$H_P = \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + V_{\text{Coul}} \quad (50)$$

$$H_I = H_{I1} + H_{I2} + H_{I1}^S \quad (51)$$

$$H_{I1} = - \sum_{\alpha} \frac{q_{\alpha}}{m_{\alpha}} \mathbf{p}_{\alpha} \cdot \mathbf{A}_{\perp}(\mathbf{r}_{\alpha}) \quad (52)$$

$$H_{I1}^S = - \sum_{\alpha} g_{\alpha} \frac{q_{\alpha}}{2m_{\alpha}} \mathbf{S}_{\alpha} \cdot \mathbf{B}(\mathbf{r}_{\alpha}) \quad (53)$$

$$H_{I2} = \sum_{\alpha} \frac{q_{\alpha}^2}{2m_{\alpha}} \mathbf{A}_{\perp}^2(\mathbf{r}_{\alpha}). \quad (54)$$

# Electric Dipole Approx.

$$r \ll \lambda$$

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} [\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{0})]^2 + V_{\text{Coul}} + \sum_j \hbar \omega_j (a_j^{\dagger} a_j + \frac{1}{2}). \quad (71)$$

$$\vec{d} = \sum_{\alpha} q_{\alpha} \vec{r}_{\alpha}$$

$$T = \exp\left[-\frac{i}{\hbar} \mathbf{d} \cdot \mathbf{A}_{\perp}(\mathbf{0})\right] = \exp\left\{\sum_j (\lambda_j^* a_j - \lambda_j a_j^{\dagger})\right\} \quad (72)$$

$$H' = THT^{\dagger}$$

$$\begin{aligned} &= \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + V_{\text{Coul}} + \varepsilon_{\text{dip}} + \sum_j \hbar \omega_j (a_j^{\dagger} a_j + \frac{1}{2}) - \\ &\quad - \mathbf{d} \cdot \sum_j \mathcal{E}_{\omega_j} [ia_j \boldsymbol{\epsilon}_j - ia_j^{\dagger} \boldsymbol{\epsilon}_j] \end{aligned} \quad (75)$$

$$\varepsilon_{\text{dip}} = \sum_j \frac{1}{2\varepsilon_0 L^3} (\boldsymbol{\epsilon}_j \cdot \mathbf{d})^2. \quad (76)$$

$$\mathbf{v}'_{\alpha} = T \mathbf{v}_{\alpha} T^{\dagger} = \frac{\mathbf{p}_{\alpha}}{m_{\alpha}} \quad (78)$$

$$H' = THT^+$$

$$= \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2m_{\alpha}} + V_{\text{Coul}} + \varepsilon_{\text{dip}} + \sum_j \hbar\omega_j \left( a_j^{\dagger} a_j + \frac{1}{2} \right) - \mathbf{d} \cdot \sum_j \mathcal{E}_{\omega_j} [i a_j \boldsymbol{\varepsilon}_j - i a_j^{\dagger} \boldsymbol{\varepsilon}_j] \quad (75)$$

$$\varepsilon_{\text{dip}} = \sum_j \frac{1}{2\varepsilon_0 L^3} (\boldsymbol{\varepsilon}_j \cdot \mathbf{d})^2. \quad (76)$$

$$\mathbf{v}'_{\alpha} = T \mathbf{v}_{\alpha} T^+ = \frac{\mathbf{p}_{\alpha}}{m_{\alpha}} \quad (78)$$

$$\begin{aligned} \mathbf{D}'(\mathbf{r})/\varepsilon_0 &= \mathbf{E}'_{\perp}(\mathbf{r}) + \frac{1}{\varepsilon_0} \mathbf{P}'_{\perp}(\mathbf{r}) \\ &= \mathbf{E}_{\perp}(\mathbf{r}) \\ &= i \sum_j \mathcal{E}_{\omega_j} (a_j \boldsymbol{\varepsilon}_j e^{i\mathbf{k}_j \cdot \mathbf{r}} - a_j^{\dagger} \boldsymbol{\varepsilon}_j e^{-i\mathbf{k}_j \cdot \mathbf{r}}). \end{aligned} \quad (89)$$

$$\begin{aligned} \mathbf{E}'_{\perp}(\mathbf{r}) &= T \mathbf{E}_{\perp}(\mathbf{r}) T^+ \\ &= \sum_j \mathcal{E}_{\omega_j} [i(a_j + \lambda_j) \boldsymbol{\varepsilon}_j e^{i\mathbf{k}_j \cdot \mathbf{r}} + \text{h.c.}] \\ &= \mathbf{E}_{\perp}(\mathbf{r}) - \frac{1}{\varepsilon_0} \mathbf{P}_{\perp}(\mathbf{r}) \end{aligned} \quad (81)$$

$$\mathbf{P}(\mathbf{r}) = \mathbf{d} \delta(\mathbf{r}) \quad (82)$$

$$H'_1 = -\mathbf{d} \cdot \mathbf{D}'(\mathbf{0})/\varepsilon_0$$

$$H'_2 = -\mathbf{d} \cdot \mathbf{E}_{\perp}(\mathbf{0})$$