The Theory of Quantum Amplifiers

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The Theory of Quantum Amplifiers

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Abstract

The theory of linear quantum amplifiers is reviewed. A simplified method using a "faked vacuum" is introduced to derive amplifier master equations. Properties of various amplifiers are summarized. The solution is derived and used as the amplifier transfer function. The Caves' theory for amplifier added noise is discussed. The properties of squeezed states are reviewed, the use of the Wigner function to describe them is advocated. Finally the amplification of squeezed states is discussed in the Wigner representation.

Prolegomena

Until quite recently the performance of most laser devices was limited by thermal noise or mechanical vibrations of the experimental set-up. Hence the numerous discussions of quantum limitations [1] were largely of an academic character except near the laser threshold where the behaviour is totally dominated by noise.

Recently, however, the technical achievements of lasers has been improved and the influence of quantum noise has acquired new interest. Especially the work on ring laser gyros has produced lasers that operate close to the quantum limit [2].

The increased use of lasers in ultra-high precision measurements, especially the work on gravitational wave detection [3], has focussed the attention on optimization criteria for laser amplifiers. Also in communication applications the ultimate performance limit may constitute an important factor in dimensioning the optical systems.

The process of linear amplification is the counterpart of linear damping, which is well known to be accompanied by unavoidable fluctuations (see Refs. [1]). If these are neglected both cases lead to similar difficulties. If we assume the gain factor to be given by $W(>0)$, one might naively assume that an amplification process can be achieved if we write for the boson annihilation operator a the Heisenberg equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}a = \frac{1}{2}Wa - \mathrm{i}\omega a. \tag{1}$$

The integration of this equation, however, leads to the commutator

$$[a(t), a^\dagger(t)] = \mathrm{e}^{Wt}, \tag{2}$$

which shows that the time evolution does not preserve unitarity.

In order to achieve a consistent description of a linear amplifier Yuen [4], Caves and others, [5] and [3], have introduced the model shown in Fig. 1. Here a is the input boson, and b is the output boson, which may be the same field at a later instant. They are connected by a linear transfer function $G^{1/2}$, but to retain unitarity we must add the influence of external noise in the form of an explicit noise operator F. We write the basic transfer relation in the form

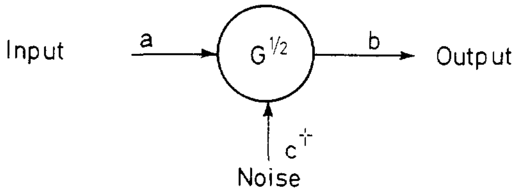

Fig. 1. This is the abstract model of a linear amplifier transforming an input boson a into an output boson b with the gain G. To ensure unitarity a noise input is needed, which can be generated by a third boson created by $c^\dagger$.

$$b = G^{1/2}a + F. \tag{3}$$

For an incoming boson, the output is described by a boson if we have

$$[b, b^\dagger] = G[a, a^\dagger] + [F, F^\dagger] = 1. \tag{4}$$

This can be satisfied if the noise operator F is related to a boson creator operator $c^\dagger$ so that

$$F = \sqrt{G-1}\, c^\dagger. \tag{5}$$

For an amplifier, $G > 1$, this choice satisfied eq. (4). For an absorber, $G < 1$, there appears a destruction operator c in place of $c^\dagger$.

If the boson fields carry information, the expectation values $\langle a\rangle$, $\langle b\rangle$ are nonvanishing, and Caves [5] defines their fluctuations as the *symmetrized* correlation functions. For *any* boson operator we then find the uncertainty relations

$$\begin{aligned}\overline{|\Delta b|^2} &= \tfrac{1}{2}\langle(b^\dagger b + bb^\dagger)\rangle - \langle b\rangle\langle b^\dagger\rangle \\ &= \tfrac{1}{2}\langle[b, b^\dagger]\rangle + \langle(b^\dagger - \langle b^\dagger\rangle)(b - \langle b\rangle)\rangle \\ &\geqslant \tfrac{1}{2}.\end{aligned} \tag{6}$$

Consequently the noise operator in eq. (3) satisfies by eq. (5) the relation

$$\begin{aligned}\overline{|\Delta F|^2} &= \tfrac{1}{2}\langle(FF^\dagger + F^\dagger F)\rangle \\ &\geqslant \tfrac{1}{2}(G-1).\end{aligned} \tag{7}$$

Caves introduces the equivalent noise factor

$$\mathscr{A} = \frac{\overline{|\Delta F|^2}}{G} \geqslant \frac{1}{2}\left(1 - \frac{1}{G}\right). \tag{8}$$

For large amplification, $G \to \infty$, this factor exceeds one half. From the basic amplifier equation (3) we find the relation

$$\begin{aligned}\overline{|\Delta b|^2} &= G\overline{|\Delta a|^2} + \overline{|\Delta F|^2} \\ &= G(\overline{|\Delta a|^2} + \mathscr{A}) \\ &\geqslant G(\tfrac{1}{2} + \tfrac{1}{2}).\end{aligned}$$

From this relation Caves concludes, that quantum mechanics extracts its due twice; one half of a photon is the minimum input fluctuation and the amplification process adds noise equivalent to one half photon at the amplifier input. The actual value of one half derives from the choise of symmetrically ordered fluctuations; if we choose the normally ordered operators, it is easy to see that the equivalent noise at the amplifier input is one photon.

Beautiful as the formal theory above may be, it leaves many questions open: By how much does the noise exceed the minimum in eq. (9), is there any amplifier that approaches the ideal operation of an equality in eq. (9), and what are the conditions minimizing the noise? To answer these questions one must construct simple models of linear amplifiers and try to derive their noise properties. By explicitly introducing the quantum dynamics of the amplifying degrees of freedom, we avoid all use of the quantum noise sources. In addition we can relate the amplifier parameters directly to the physical mechanism in operation. This is the first step in an effort to understand how to approach the technical realizations of the ideal performance of the idealized models. These questions form the subject presented in the present lectures.

In section 1 we discuss the theory of linear amplification and the simplest models of amplifier systems. In section 2 we discuss the solutions of the amplifier equations and their properties. In section 3 we discuss how to describe the squeezed states, which recently have acquired a lot of interest. For easy access and completeness some of the main mathematical results needed are given in the Appendix. Some of these are derived but for most of them the reader is referred to the references.

1. How to build an amplifier

1.1. *Quantum theory of absorbers*

The quantum mechanical procedure for obtaining exponential decay is well known; the theory was first introduced by Weisskopf and Wigner [6]. It has later been developed into a systematic way to treat irreversible phenomena [7]. Here we go through the most elementary argument to illustrate the main physical ideas; no rigour is attempted.

We consider an isolated state $|1\rangle$ interacting with a range of continuum states $|k\rangle$ as shown in Fig. 2. The Hamiltonian is written

$$H = \hbar[\omega|1\rangle\langle 1| + \sum_k \Omega_k|k\rangle\langle k| + \sum_k \lambda_k(|k\rangle\langle 1| + |1\rangle\langle k|)] \tag{10}$$

and the state vector is taken in the form

$$|\psi\rangle = \left[a|1\rangle + \sum_k C_k|k\rangle\right]\mathrm{e}^{-\mathrm{i}\omega t}. \tag{11}$$

The explicit time dependence is introduced to eliminate rapidly varying components. After Laplace transforming we obtain the equations

$$\mathrm{i}s\tilde{a} - \mathrm{i}a(0) = \sum_k \lambda_k \tilde{C}_k \tag{12}$$

$$\mathrm{i}s\tilde{C}_k = (\Omega_k - \omega)\tilde{C}_k + \lambda_k\tilde{a}, \tag{13}$$

which for the amplitude of the isolated state gives the equation

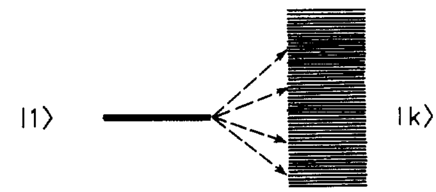

Fig. 2. In the Wigner–Weisskopf model of atomic decay an isolated state $|1\rangle$ is coupled to a continuum $|k\rangle$.

$$\mathrm{i}s\tilde{a} - \mathrm{i}a(0) = -\sum_k \frac{\lambda_k^2}{\Omega_k - \omega - \mathrm{i}s}\tilde{a}. \tag{14}$$

When we want to consider the decay of the isolated state, we can assume that the continuum states are empty at the initial time.

The behaviour for long times is determined by the behaviour of the Laplace transform for small values of s. Inside the sum, assuming the continuum limit, we can let s go to zero, and the resulting equation of motion becomes

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= \mathrm{i}\int \mathrm{d}\Omega\,\frac{\lambda^2(\Omega)D(\Omega)}{\Omega - \omega - \mathrm{i}s}\,a \\ &= \mathrm{i}\Delta\omega a - \tfrac{1}{2}\Gamma a,\end{aligned} \tag{15}$$

where $D(\Omega)$ is the density of Ω-states. Here the decay rate is the ordinary Weisskopf–Wigner result

$$\Gamma = 2\pi\lambda^2 D(\omega) = \frac{2\pi}{\hbar}|\langle 1|H|k\rangle|^2\frac{D(\omega)}{\hbar}, \tag{16}$$

which is seen to agree with the "Golden Rule" of perturbation theory because

$$D(E) = \frac{D(\Omega)}{\hbar} \tag{17}$$

is the density of energy states. The frequency shift is given by

$$\Delta\omega = \mathscr{P}\int \mathrm{d}\Omega\,\frac{\lambda^2(\Omega)D(\Omega)}{\Omega - \omega} \tag{18}$$

This is seen to depend on the combination $\lambda^2 D$ in a more essential way. The integral is often divergent, equally often set equal to zero, but it can, in fact, be given any value by a slight adjustment of the tails of the function $\lambda^2 D$.

In order to obtain the decay rate (16) we need a continuum of states near the resonance position $\omega \approx \Omega$. Once this is satisfied, the density of states times the coupling strength at one frequency determines the decay; their functional dependence will affect only the shift. Thus we may conclude that any mechanism leading to a finite density of states will be adequate. We will try to present a modified model where there is no real continuum but a *faked one*. This method was first introduced by the present author in a different context [8].

Instead of the range of oscillators in eqs. (10) and (12) we take only one

$$\mathrm{i}\,\dot{a} = \lambda c. \tag{19}$$

In contrast to the previous case, we now assume that the single oscillator is heavily damped, and its equation of motion is written

$$\mathrm{i}\,\dot{c} = (\Omega - \omega)c - \mathrm{i}\kappa c + \lambda a. \tag{20}$$

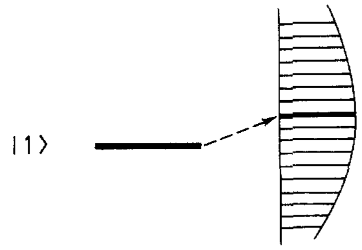

Fig. 3. As far as the decay is concerned the Wigner–Weisskopf model can be replaced by one single strongly dampled state; a "faked continuum".

The situation is depicted in Fig. 3, where we can see how the Lorentzian line of the damped oscillator offers a density of states to the isolated state thereby faking the effect of the continuum.

To solve this problem we do not need a Laplace transform; the conventional adiabatic elimination procedure suffices [8]. We assume the damping κ to be strong enough to bring the damped mode into its instantaneous value imposed by the isolated mode. We solve (20) and insert the result

$$c = -\frac{\lambda}{\Omega - \omega - i\kappa}\, a \tag{21}$$

into eq. (19). This gives an equation of motion identical with eq. (15). The shift is given by

$$\Delta\omega = \frac{\lambda^2(\Omega - \omega)}{(\Omega - \omega)^2 + \kappa^2}, \tag{22}$$

which as before is model dependent but lacks interest. The width becomes

$$\Gamma = 2\pi\lambda^2 \frac{\kappa/\pi}{(\Omega - \omega)^2 + \kappa^2}, \tag{23}$$

which in the limit of large damping becomes

$$\Gamma = 2\pi\lambda^2 D(\omega); \tag{24}$$

here the density of states near the resonance frequency is correctly given by

$$D(\omega) = \frac{1}{\pi\kappa}. \tag{25}$$

From this simple calculation we can conclude that, when we want to calculate the decay, it does not matter how we introduce the smooth density of states near the resonance frequency. Only the frequency shift will be strongly dependent on the details of the interaction model.

1.2. *The amplifier master equation*

An amplifier breaks the time reversal symmetry inherent in the formulation of quantum mechanics. The amplification process is just the inverse of an attenuator or damping. Damping is the ordinary cause of irreversibility, and it has been discussed much both within the classical theory and in quantum theory. We known the physically motivated approximations required for the introduction of irreversibility.

The master equation for a quantum system was first derived in the context of magnetic resonance by Wangsness, Bloch and Redfield [9]. For optical resonances these techniques were utilized by Weidlich, Haken and Lax [10].

A system of interest is perturbed by a coupling to a bath through the interaction picture potential $V(t)$. A perturbative iteration to second order gives for the reduced density matrix of the system of interest the result

$$\dot{\varrho}_s(t) = -\frac{1}{\hbar^2}\int_0^t dt' \, [\langle V(t)V(t')\rangle_B \varrho_s(t') - \langle V(t)\varrho_s(t')V(t')\rangle_B + \text{h.a.}]; \tag{26}$$

see for instance [11]. The subscript **B** on the averages denotes an average over the degrees of freedom of the bath. The result (26) assumes a weak interaction (the Born approximation). If the bath is complicated enough (i.e. has got many degrees of freedom) the bath correlations decay within a short time. Then we can assume that the system possesses no memory (Markov property) and inside the integration make the replacement

$$\varrho_s(t') \to \varrho_s(t), \tag{27}$$

and after that we may extend the integration limit in eq. (26) to infinity, because the value of the integral is determined by small time differences. This is the standard procedure for deriving Born-Markov master equations for dissipative systems.

The master equation for an amplifier can be derived in exactly the same way, but we have to arrange the state of the bath in such a way that it can feed energy into the system of interest instead of sucking energy from it. Unless we account for the depletion of the energy of the bath, the time evolution of an amplifier leads to an explosive growth, which finally violates unitarity. In laser models the growth is arrested by the finite energy supplied by the pumping; the laser saturates. We can, however, assume the pumping to constitute a heat bath over times such that the linear dependence prevails. This is the region of linear amplification.

We could obtain the amplifier equation by a straightfoward application of eq. (26), but it is instructive to carry out the same derivation using the faked continuum introduced in the previous Section. In order to proceed we assume that we go to the interaction picture, and at resonance the coupling is given by a time-independent operator of the form

$$H = \hbar g(\Gamma^\dagger a + \Gamma a^\dagger), \tag{28}$$

where Γ is an unspecified operator acting on the degrees of freedom of the bath; for examples see Section 1.3. The mode where amplification is to take place is described by the boson creation operator $a^\dagger$. We introduce the density matrix reduced to the space of the boson mode

$$P = \mathrm{Tr}_B \varrho, \tag{29}$$

and find from eq. (28) its equation of motion

$$i\dot{P} = g(a\varphi^\dagger + a^\dagger\varphi - \varphi^\dagger a - \varphi a^\dagger), \tag{30}$$

with

$$\varphi^\dagger = \mathrm{Tr}_B\, \Gamma^\dagger \varrho = \mathrm{Tr}_B\, \varrho\Gamma^\dagger \tag{31}$$

$$\varphi = (\varphi^\dagger)^\dagger = \mathrm{Tr}_B\, \Gamma\varrho. \tag{32}$$

Next we write the equations of motion for the quantities φ and $\varphi^\dagger$, which are still operators on the boson degrees of freedom. As the bath is supposed to offer the bosons a continuum of states, we make all bath variables overdamped by adding a damping γ. It would be straightfoward to deduce this from a model too; the technique for introducing damping was described above. The assumption that the operator φ is damped in the same way as the bath operator Γ is closely related to the so called quantum regression theorem [12]. The ensuing equation is

$$\begin{aligned} \mathrm{i}\dot{\varphi} &= -\mathrm{i}\gamma\varphi + \mathrm{Tr_B}\,\Gamma[H, \varrho] \\ &= -\mathrm{i}\gamma\varphi + g[a\langle\Gamma\Gamma^\dagger\rangle_\mathrm{B} + a^\dagger\langle\Gamma^2\rangle_\mathrm{B} \\ &\quad -\langle\Gamma^\dagger\Gamma\rangle_\mathrm{B} a - \langle\Gamma^2\rangle_\mathrm{B} a^\dagger]. \end{aligned} \tag{33}$$

The time development of the reduced density matrix P in eq. (29) is already of first order with respect to the coupling constant g. In the weak perturbation approximation we need to go only to second order in the coupling, and hence we need to solve eq. (33) only to first order. Consequently we can replace the density matrix ϱ on the right-hand side of eq. (33) by its unperturbed value $\varrho_\mathrm{B}^0 P$ which allows the decouplings

$$\begin{aligned} \langle\Gamma\Gamma^\dagger\rangle_\mathrm{B} &= \mathrm{Tr_B}\,\Gamma\Gamma^\dagger\varrho \\ &= \mathrm{Tr_B}\,(\Gamma\Gamma^\dagger\varrho_\mathrm{B}^0)\,P = \langle\Gamma\Gamma^\dagger\rangle\,P. \end{aligned} \tag{34}$$

Assuming it possible to eliminate the quantity φ adiabatically and adopting the decoupling procedure in eq. (34) we obtain from eq. (33) the result

$$\varphi = -\frac{\mathrm{i}g}{\gamma}[\langle\Gamma\Gamma^\dagger\rangle aP + \langle\Gamma^2\rangle\,[a^\dagger, P] - \langle\Gamma^\dagger\Gamma\rangle Pa]. \tag{35}$$

When this is inserted into the equation of motion for the reduced density matrix (30), we obtain

$$\begin{aligned} \dot{P} &= -\frac{g^2}{\gamma}[\langle\Gamma^{\dagger 2}\rangle[a, [a, P]] + \langle\Gamma^2\rangle\,[a^\dagger, [a^\dagger, P]] \\ &\quad + \langle\Gamma\Gamma^\dagger\rangle(a^\dagger aP - 2aPa^\dagger + Pa^\dagger a) \\ &\quad + \langle\Gamma^\dagger\Gamma\rangle(aa^\dagger P - 2a^\dagger Pa + Paa^\dagger)]. \end{aligned} \tag{36}$$

This is the master equation for the linear regime of a boson mode imbedded in a continuum; depending on its properties it describes amplification or damping.

How does the result obtained relate to the general theory described by eq. (26)? An insertion of the perturbation (28) into the general formula (26) would have produced correlation functions of the type

$$K(t) = \langle\Gamma(t)\Gamma^\dagger(0)\rangle. \tag{37}$$

In the Markov approximation these would have been taken to decay exponentially with some rate γ and the coefficients in eq. (26) would have become

$$\begin{aligned} g^2\int_0^\infty \langle\Gamma(t)\Gamma^\dagger(0)\rangle\,\mathrm{d}t &= g^2\int_0^\infty \langle\Gamma\Gamma^\dagger\rangle\,\mathrm{e}^{-\gamma t}\mathrm{d}t \\ &= \frac{g^2}{\gamma}\langle\Gamma\Gamma^\dagger\rangle. \end{aligned} \tag{38}$$

This is exactly the result we obtained by our simplified faked continuum; in both cases all possible frequency shifts have been neglected. If we write the differential gain coefficient from the master equation (36) we find

$$W = \frac{2g^2}{\gamma}\langle[\Gamma^\dagger, \Gamma]\rangle, \tag{39}$$

which relates our treatment directly to the general linear response theory.

1.3. *Various amplifier configurations*

The simplest standard amplifier configuration consists of an assembly of two-level atoms with an inverted population. This is the standard model of a laser; its linear operation regime describes an amplifier. The interaction Hamiltonian is

$$H = \hbar\Big(g\sum_i(\sigma_i^+ a + \sigma_i^- a^\dagger\Big), \tag{40}$$

and hence the bath operators in eq. (28) are

$$\Gamma = \sum_i \sigma_i^-; \qquad \Gamma^\dagger = \sum_i \sigma_i^+. \tag{41}$$

With these we can directly evaluate the coefficients in eq. (36) and find

$$\begin{aligned} \langle\Gamma^2\rangle &= \sum_{ij}\langle\sigma_i^-\sigma_j^-\rangle = \sum_i\langle\sigma_i^{-2}\rangle = 0 \\ \langle\Gamma^{\dagger 2}\rangle &= 0 \end{aligned} \tag{42}$$

$$\begin{aligned} \langle\Gamma^\dagger\Gamma\rangle &= \sum_{ij}\langle\sigma_i^+\sigma_j^-\rangle = \sum_i\langle\sigma_i^+\sigma_i^-\rangle \\ &= \sum_i n_{2i} = N_2 = \gamma r_2 \end{aligned} \tag{43}$$

$$\langle\Gamma\Gamma^\dagger\rangle = \sum_{ij}\langle\sigma_i^-\sigma_j^+\rangle = N_1 = \gamma r_1. \tag{44}$$

Here N_i is the average population on level i, and r_i is the corresponding pumping rate into level i [13]. We find the master equation of the well known form

$$\begin{aligned} \dot{P} &= -\frac{A}{2}(aa^\dagger P - 2a^\dagger Pa + Paa^\dagger) \\ &\quad - \frac{C}{2}(a^\dagger aP - 2aPa^\dagger + Pa^\dagger a), \end{aligned} \tag{45}$$

with

$$A = \frac{2g^2}{\gamma^2}r_2 \tag{46}$$

$$C = \frac{2g^2}{\gamma^2}r_1. \tag{47}$$

The population inversion can be represented by an effective temperature T,

$$\frac{N_2}{N_1} = \frac{r_2}{r_1} = \mathrm{e}^{-\hbar\omega/kT}, \tag{48}$$

which for the amplifier case is found to be negative.

An amplifier configuration which has been discussed extensively in the literature is the inverted oscillator amplifier [14]. This is illustrated in Fig. 4, where the creation of one bath photon by $b^\dagger$ is accompanied by the creation of a photon

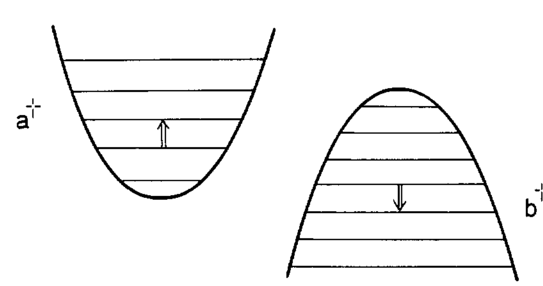

Fig. 4. The most common amplifier model is the one where the mode to be amplified, $a^\dagger$, derives its energy from an "inverted oscillator" whose quanta are created by $b^\dagger$.

of the field to be amplified. The interaction Hamiltonian is of the form

$$H = \hbar g(ab + a^\dagger b^\dagger). \tag{49}$$

The popularity of this model derives from the fact, that the bilinear Hamiltonian can be solved exactly, and the linear amplifier regime can be extracted. If the bath mode is in a thermal state we have

$$\langle b^2 \rangle = \langle b^{\dagger 2} \rangle = 0$$
$$\langle bb^\dagger \rangle = \langle b^\dagger b \rangle + 1 = n_B + 1, \tag{50}$$

and a master equation of the form (45) follows. The coefficients are given by

$$A \propto n_B + 1 \tag{51}$$
$$C \propto n_B. \tag{52}$$

This is the amplifier model used most in the literature.

One may question the possibility to realize the inverted oscillator amplifier physically. It may, however, be regarded as the limiting case of a parametric amplifier, when all pump fields except one idler mode are strong enough to be described by classical fields. A particularly interesting application of this model has been provided by Haake and Glauber in their treatment of the initial stages of the superfluorescence process [15]. A fully excited assembly of two-level atoms displays a ladder of equally spaced levels, which extends downwards for many steps. As long as the process is not saturating these levels behave exactly as an inverted oscillator, and this fact has been utilized by Haake and Glauber to investigate the quantum properties of the onset of superfluorescence.

If we allow just the simple exchange of a boson of interest for a bath photon, the Hamiltonian is

$$H = \hbar g(ab^\dagger + a^\dagger b). \tag{53}$$

It is easy to see that the resulting master equation is of the form (45) with the coefficients

$$A \propto n_B \qquad C \propto n_B + 1. \tag{54}$$

In this case the model can only provide damping and no amplification is possible; see next section.

It is possible to investigate more complex situations. The four-wave mixing Hamiltonian

$$H = \hbar g(a_1^\dagger a_2^\dagger a_3 a_4 + a_4^\dagger a_3^\dagger a_2 a_1) \tag{55}$$

describes a parametric process of the type shown in Fig. 5. This also provides a region of linear gain, but it would take us too far to consider this case in detail here.

2. Properties of amplifiers

2.1. *Solution of the amplifier and absorber equations*

In most cases of interest it suffices to consider the single-mode linear master equation to be of the type (45). If $A > C$ it describes an amplifier, if $A < C$ it describes an absorber. To obtain a picture of its properties we choose to use the Glauber–Sudarshan representation of the density matrix

$$\varrho = \int P(z)|z\rangle \, d^2z \langle z|, \tag{56}$$

which inserted into eq. (45) gives the Fokker–Planck equation

$$\dot{P} = \frac{\partial}{\partial z}[(i\omega - \tfrac{1}{2}W)zP] - \frac{\partial}{\partial z^*}[(i\omega + \frac{1}{2}W)z^*P] + A\frac{\partial^2}{\partial z \partial z^*}P, \tag{57}$$

where the linear gain (or damping) is

$$W = A - C, \tag{58}$$

and the diffusion derives entirely from the amplifying process A; this is understandable because the amplification process is the one accompanied by spontaneous emission noise. For mathematical details see the Appendix.

For the amplitude of the field we immediately obtain the equation

$$\frac{d}{dt}\langle z \rangle = \tfrac{1}{2}W\langle z \rangle - i\omega\langle z \rangle, \tag{59}$$

with the solution

$$\langle z(t) \rangle = G^{1/2} e^{-i\omega t} \langle z(0) \rangle, \tag{60}$$

where the gain has been denoted by

$$G(t) = e^{Wt}. \tag{61}$$

The Glauber–Sudarshan function can directly be used to calculated normally ordered expectation values, and hence we can calculate the mean photon number

$$\langle n \rangle \equiv \langle a^\dagger a \rangle = \langle z^*(t) z(t) \rangle \tag{62}$$

in the form

$$\frac{d}{dt}\langle n \rangle = W\langle n \rangle + A = A(\langle n \rangle + 1) - C\langle n \rangle. \tag{63}$$

In this equation we can directly see how the coefficient $W = A - C$ describes the linear growth (gain) of the energy in the mode, whereas the diffusion term A represents the noise provided by the spontaneous emission; this is present even if the mode energy is initially zero.

The equation (57) can be solved exactly; the solution which approaches a delta function for the initial time is given by

$$P(z, t|z_0) = \frac{1}{\pi m(t)} \times \exp\left[-\frac{(z - G^{1/2}e^{-i\omega t}z_0)(z^* - G^{1/2}e^{i\omega t}z_0^*)}{m(t)}\right], \tag{64}$$

where the time dependent width is given by

$$m(t) = \frac{A}{W}[G(t) - 1]. \tag{65}$$

For a long time, $t \to \infty$, the gain growth to infinity, but so does the width. This fact is related to the Caves limit of the amplifier, see next Section.

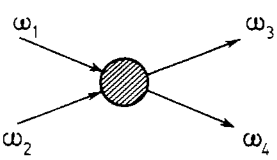

Fig. 5. The general four-wave parametric process, where the linear gain of one mode, ω_4 say, can be described by a linear amplifier master equation.

If the master equation describes an absorber, we have instead of eq. (65)

$$m(t) = \frac{A}{C - A}[1 - \eta], \tag{66}$$

where the damping factor is

$$\eta = e^{-(C-A)t} < 1. \tag{67}$$

For long times this approaches a finite limit; the absorber can only add a limited amount of noise. The difference between the behaviour of the amplifier and the absorber is illustrated in Figs. 6(a) and 6(b). These results were given originally in Refs. [16].

2.2. *The amplifier added noise*

The solution (64) which reduces to a delta function can be used as an integral kernel to calculate the transformation of any incoming P-function, $P_0(z)$, by the amplifier; it is the amplifier *transfer function*. After passing the amplifier the signal has got its state described by

$$P_{\rm out}(z) = \int \frac{dz_0}{\pi m(t)} \times \exp\left[- \frac{(z - G^{1/2}e^{-i\omega t}z_0)(z^* - G^{1/2}e^{i\omega t}z_0^*)}{m(t)} \right] P_0(z_0). \tag{68}$$

This form has been used to discuss the statistical properties of the amplified signal by Mandel et al. [17].

In particular we can calculate the symmetrically ordered fluctuation (6) as defined by Caves [5]. To do this we must relate the desired quantity to the normally ordered expectation values, which follows directly from the form (68):

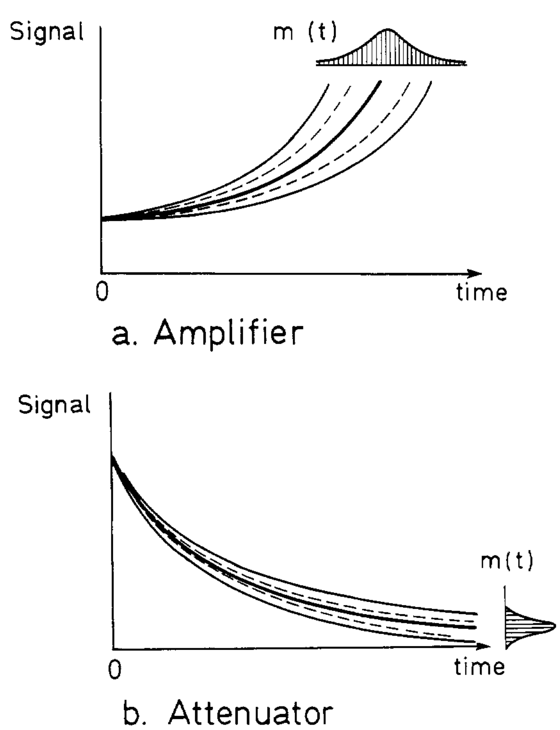

Fig. 6. In (a) it is shown how the linearly amplified signal grows exponentially and broadens without limit. In (b) the exponentially attentuated signal approaches a finite width for long times.

$$\begin{aligned}\langle \Delta a^\dagger \Delta a \rangle_{\rm out} &\equiv \langle a^\dagger a \rangle - \langle a^\dagger \rangle \langle a \rangle \\ &= [G\langle \Delta a^\dagger \Delta a \rangle_{\rm in} + m(t)].\end{aligned} \tag{69}$$

Noting the relation

$$\langle \Delta a^\dagger \Delta a \rangle = \overline{|\Delta a|^2} - \tfrac{1}{2}, \tag{70}$$

we can write

$$\begin{aligned}\overline{|\Delta a|^2_{\rm out}} &= \tfrac{1}{2}[\langle \Delta a^\dagger \Delta a \rangle + \langle \Delta a \Delta a^\dagger \rangle] \\ &= G\langle \Delta a^\dagger \Delta a \rangle_{\rm in} + m(t) + \tfrac{1}{2} \\ &= G\overline{|\Delta a|^2_{\rm in}} + m(t) - \tfrac{1}{2}(G - 1).\end{aligned} \tag{71}$$

The term

$$m(t) - \tfrac{1}{2}(G - 1) = \frac{1}{2}\left(\frac{A + C}{A - C}\right)(G - 1) \tag{72}$$

is the noise added by the amplifier. If we use the noise factor (8) introduced by Caves we find

$$\mathscr{A} = \frac{1}{2}\left(\frac{A + C}{A - C}\right)\left(1 - \frac{1}{G}\right) \geqslant \frac{1}{2}\left(1 - \frac{1}{G}\right); \tag{73}$$

for an infinite gain the Caves' inequality (9) is regained.

If we introduce the excess noise factor ε by writing

$$\frac{1}{2}\left(\frac{A + C}{A - C}\right) = \tfrac{1}{2} + \varepsilon, \tag{74}$$

we find for a two-level amplifier, from eqs. (46)–(48), the result

$$\varepsilon = \frac{r_1}{r_2 - r_1} = \frac{1}{e^{-\hbar\omega/kT} - 1}\ (T < 0). \tag{75}$$

For the inverted oscillator amplifier with eqs. (51) and (52) we obtain

$$\varepsilon = n_{\rm B}. \tag{76}$$

In both cases, when the effective temperature goes to zero, $T \to 0$ the excess noise disappears. This agrees with our intuitive expectations.

A similar calculation for the absorber shows that the noise relationship betwen input and output becomes

$$\overline{|\Delta a|^2_{\rm out}} = \eta\overline{|\Delta a^2_{\rm in}} + \frac{1}{2}\frac{C + A}{C - A}(1 - \eta). \tag{77}$$

The noise is at its minimum when there is no amplification, $A = 0$, as one might expect. For a large damping, $\eta \to 0$, the absorber only produces half a photon of noise at the output; the input noise is attenuated out, at the same rate as the original signal. These results are discussed further in Refs. [16].

3. The amplification of squeezed states

3.1. *The description of squeezed states*

The generally accepted quantization procedure of the electromagnetic field begins by transforming Maxwell's theory into a theory for harmonically oscillating eigenmodes. Each mode can then be quantized in a routine fashion, and for each mode the Hamiltonian can be written

$$\begin{aligned}H &= \tfrac{1}{2}(p^2 + \omega^2 q^2) \\ &= \omega(a^\dagger a + \tfrac{1}{2}),\end{aligned} \tag{78}$$

where the operators are related by

$$a = \sqrt{\frac{\omega}{2}}\left(q + \mathrm{i}\,\frac{p}{\omega}\right), \tag{79}$$

and satisfy the familiar commutation relations

$$[q, p] = \mathrm{i} \qquad [a, a^\dagger] = 1. \tag{80}$$

It is also useful to introduce the dimensionless variables

$$u = \sqrt{\omega}\, q \qquad \text{and} \qquad v = p/\sqrt{\omega}, \tag{81}$$

which obey the same commutation relations as p and q.

The ordinary coherent states can be generated by the linear transformation

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}; \tag{82}$$

see the discussion in the Appendix.

If we write the parameter α in the form

$$\begin{aligned}\alpha &= \frac{1}{\sqrt{2}}(\bar{u} + \mathrm{i}\bar{v}),\\ &= \sqrt{\frac{\omega}{2}}\left(\bar{q} + \mathrm{i}\,\frac{\bar{p}}{\omega}\right),\end{aligned} \tag{83}$$

we find that the transformation shifts the center of the q and p variables to $\bar{q}$ and $\bar{p}$ respectively.

The commutation relations (80) imply the ordinary uncertainty relation

$$\Delta p \Delta q \geqslant \tfrac{1}{2}, \qquad \Delta u \Delta v \geqslant \tfrac{1}{2}. \tag{84}$$

If the uncertainty minimum is distributed symmetrically between the two components we realize the quantum limit of uncertainty:

$$\Delta u = \Delta v = \frac{1}{\sqrt{2}}. \tag{85}$$

It is, however, also possible to satisfy the uncertainty relation by an unequal distribution of the quantum fluctuations

$$\begin{aligned}\sqrt{\omega}\Delta q &= \Delta u = \frac{s}{\sqrt{2}};\\ \frac{\Delta p}{\sqrt{\omega}} &= \Delta v = \frac{1}{\sqrt{2}\,s},\end{aligned} \tag{86}$$

where $s \neq 1$. These states are called *squeezed*, see Ref. [18]. They have attracted great interest recently, because if they can be realized they may be utilized for high sensitivity detection and communication purposes. In Fig. 7(a) the difference between symmetric and squeezed minimum uncertainty states is illustrated.

The squeezed states can be most easily realized by the linear transformation

$$S(\zeta) = \mathrm{e}^{\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)}, \tag{87}$$

where ζ is an arbitrary complex number. If we write this number in the form

$$\zeta = r\,\mathrm{e}^{\mathrm{i}\varphi} \tag{88}$$

and perform the canonical transformation

$$a \to \mathrm{e}^{-\mathrm{i}\varphi/2} a, \qquad a^\dagger \to \mathrm{e}^{\mathrm{i}\varphi/2} a^\dagger \tag{89}$$

(gauge transformation of the first kind) we obtain the squeeze operator in the q-representation

$$S(r) = \mathrm{e}^{\frac{1}{2} r (a^{\dagger 2} - a^2)} = \mathrm{e}^{-r/2}\,\mathrm{e}^{-rq(\partial/\partial q)}. \tag{90}$$

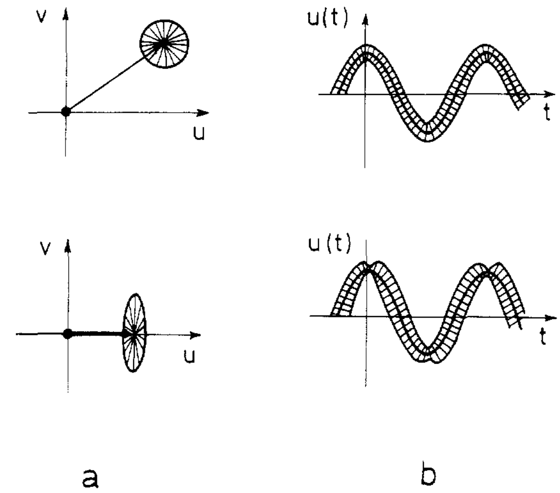

Fig. 7. In (a) we show at the top a coherent state with its uncertainties evenly distributed between the u- and v-components. At the bottom is shown a squeezed state with the uncertainty in v exceeding that in u. In (b) the corresponding time evolution is shown. Only the coherent state preserves its width while oscillating whereas the squeezed state displays a pulsating width, with the pulsations in the u- and v-components in opposite phases.

The transformation (89) is just a simple rotation of the phase in the complex plane where we represent the amplitude of the electromagnetic field oscillators.

We introduce the conventional squeezing factor [18],

$$s = \mathrm{e}^r, \tag{91}$$

and find easily that

$$\begin{aligned}S(r)f(q) &= \frac{1}{\sqrt{s}}\exp\left[-rq\,\frac{\partial}{\partial q}\right]f(q)\\ &= \frac{1}{\sqrt{s}}\exp\left[-r\,\frac{\partial}{\partial \log q}\right]f(\mathrm{e}^{\log q})\\ &= \frac{1}{\sqrt{s}}f[\exp(\log q - r)] = \frac{1}{\sqrt{s}}f\left(\frac{q}{s}\right),\end{aligned} \tag{92}$$

which holds for any function $f(q)$. If we take the ground state wave function for the harmonic oscillator in the q-representation

$$\langle q|0\rangle = \frac{1}{\pi^{1/4}}\,\mathrm{e}^{-\omega q^2/2}$$

and apply the shift operator (82) and the squeezing operator (92) we find the oscillator state

$$\langle q|D(\alpha)S(r)|0\rangle \propto \mathrm{e}^{-\omega(q-\bar{q})^2/2s^2}\,\mathrm{e}^{\mathrm{i}\bar{p}q}, \tag{94}$$

which is the wave packet solution usually given for the oscillator; see, e.g. Ref. [19]. How to technically achieve such states is not yet quite clear; see the discussion in Ref. [18].

I have elsewhere [20] argued that the natural description of the squeezed state is the Wigner function of the harmonic motion [21]. Using the variables (81) we can define this as

$$W(u, v) = \frac{1}{\pi}\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}vy}\,\langle u + \tfrac{1}{2}y|\varrho|u - \tfrac{1}{2}y\rangle. \tag{95}$$

When the squeezed state (94) is inserted into the definition (95) we obtain the corresponding Wigner function

$$W(u, v) = \frac{1}{\pi} e^{-(u-\bar{u})^2/s^2} e^{-(v-\bar{v})^2 s^2}. \quad (96)$$

From this we can immediately see, that integrating over $p = \sqrt{\omega}v$ or $q = u/\sqrt{\omega}$ we obtain the correct probability distribution in the variable q or p respectively. The widths of the two distributions are

$$\Delta u = \frac{s}{\sqrt{2}}; \qquad \Delta v = \frac{1}{\sqrt{2}s}, \quad (97)$$

which still satisfy the minimum uncertainty product (84), but unlike in the ordinary case, they here both have a meaning simultaneously.

The Wigner function equation of motion is derived to be [21]

$$\frac{\partial W}{\partial t} + \omega v \frac{\partial W}{\partial u} = \frac{1}{\mathrm{i}}\left[V\left(u + \frac{\mathrm{i}}{2}\frac{\partial}{\partial v}\right) - V\left(u - \frac{\mathrm{i}}{2}\frac{\partial}{\partial v}\right)\right]$$
$$= \frac{\partial V}{\partial u}\frac{\partial W}{\partial v} + \dots, \quad (98)$$

which for the harmonic motion becomes

$$\frac{\partial W}{\partial t} + \omega v \frac{\partial W}{\partial u} - \omega u \frac{\partial W}{\partial v} = 0. \quad (99)$$

This can easily be integrated to give

$$W(u, v) = C \exp\left[-\frac{(u\cos\omega t - v\sin\omega t - \bar{u})^2}{s^2}\right]$$
$$\times \exp[-(v - \cos\omega t + u\sin\omega t - \bar{v})^2 s^2]. \quad (100)$$

When we integrate out the momentum variable we obtain the result

$$\Phi(q) = \int W(\sqrt{\omega}q, p/\sqrt{\omega})\,\mathrm{d}p$$
$$= \frac{1}{\sqrt{\pi\sigma^2(q)}} \exp\left[-\frac{(q - Q(t))^2}{\sigma^2(t)}\right], \quad (101)$$

where the center-of-mass moves according to

$$Q(t) = \bar{q}\cos\omega t + \frac{\bar{p}}{\omega}\sin\omega t, \quad (102)$$

and the width develops as

$$\sigma^2(t) = \frac{s^2}{\omega}\cos^2\omega t + \frac{1}{\omega s^2}\sin^2\omega t$$
$$= 2\Delta q^2\cos^2\omega t + 2\frac{\Delta p^2}{\omega^2}\sin^2\omega t. \quad (103)$$

This describes a wave packet following the classical trajectory (102) but publsating in width according to eq. (103). The time development is illustrated in Fig. 8, which corresponds to the picture conventionally given for a squeezed state; cf. e.g. Fig. 7b. In the special case of symmetric uncertainty, $s = 1$, we obtain the constant width of the displaced ground state

$$\sigma^2 = \frac{1}{\omega} \quad (104)$$

first introduced by Schrödinger [22].

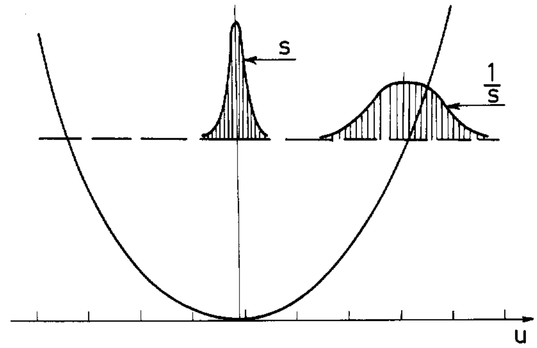

Fig. 8. The squeezed state is described by an oscillating wave packet in the harmonic potential. The width pulsates between s and s^{-1}; the picture is drawn for $s < 1$.

In conclusion I claim that the squeezed states are most easily expressed as Wigner functions; the Glauber–Sudarshan P-function does not exist for these states.

3.2. *The Wigner function master equation*

When we add the amplifier terms from eq. (45) to the ordinary density matrix equation of motion we obtain

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -\mathrm{i}[H, \varrho] - \frac{A}{2}(aa^\dagger\varrho - 2a^\dagger\varrho a + \varrho aa^\dagger)$$
$$- \frac{C}{2}(a^\dagger a\varrho - 2a\varrho a^\dagger + \varrho a^\dagger a). \quad (105)$$

With the harmonic oscillator Hamiltonian (78) and the Wigner function (95) we obtain the master equation

$$\frac{\partial W}{\partial t} + \omega v\frac{\partial W}{\partial u} - \omega v\frac{\partial W}{\partial v} = \frac{A - C}{2}\left(\frac{\partial}{\partial u}u + \frac{\partial}{\partial v}v\right)W$$
$$+ \frac{A + C}{4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)W. \quad (106)$$

This is derived in some detail in the Appendix A.3.

The Wigner function which reduces to the delta function is

$$W_{\mathrm{T}}(u, v, t\,|\,u_0, v_0) = \frac{1}{\pi m_G}$$
$$\times \exp\left[-\frac{(\bar{u} - G^{1/2}u_0)^2 + (\bar{v} - G^{1/2}v_0)^2}{m_G}\right]$$
$$G = e^{(A-C)t}$$
$$m_G = \frac{A + C}{A - C}(G - 1), \quad (107)$$

where the rotating coordinates are given by

$$\bar{u} = u\cos\omega t - v\sin\omega t$$
$$\bar{v} = v\cos\omega t + u\sin\omega t, \quad (108)$$

because of the motion derived from the free Hamiltonian. This solution can, similarly to the solution (64), be used as the transfer function of the amplifier.

3.3. *The amplification of squeezed states*

We assume the input to the amplifier to be the squeezed state described by the Wigner function (96). After being transmitted through the amplifier the state at the output is given by

$$W_{\rm out}(u, v, t) = \int W_{\rm T}(u, v, t|u_0, v_0) \frac{1}{\pi} e^{-(u_0-u_i)^2/s^2} \times e^{-s^2(v_0-v_i)^2} dv_0 du_0$$

$$= \frac{\exp\left[-\frac{(\bar{u}-G^{1/2}u_i)^2}{m_G+Gs^2} - \frac{(\bar{v}-G^{1/2}v_i)^2}{m_G+G/s^2}\right]}{\pi[(m_G+Gs^2)(m_G+G/s^2)]^{1/2}}. \quad (109)$$

We can obtain the amplifier noise added to the two components directy by reading off the widths of the two components from the results (109). We obtain

$$(\Delta u^2)_{\rm out} = G(\Delta u^2)_{\rm in} + \frac{1}{2}\left(\frac{A+C}{A-C}\right)(G-1)$$

$$(\Delta v^2)_{\rm out} = G(\Delta v^2)_{\rm in} + \frac{1}{2}\left(\frac{A+C}{A-C}\right)(G-1). \quad (110)$$

Because the symmetrically ordered expectation values can be calculated directly from the Wigner function, we can obtain the Caves' correlation function (8) at the output directly by adding the two components of eq. (110)

$$\overline{(\Delta a)^2_{\rm out}} = \tfrac{1}{2}(\overline{(\Delta u)^2_{\rm out}} + \overline{(\Delta v)^2_{\rm out}})$$

$$= G\overline{|\Delta a|^2_{\rm in}} + \frac{1}{2}\left(\frac{A+C}{A-C}\right)(G-1). \quad (111)$$

This agrees with the result we obtained earler (71).

The results (110) was earlier obtained by Friberg and Mandel [17], and a similar result by Loudon and Shephard [23].

If the input is squeezed, $s < 1$, the output can remain squeezed only if

$$G\frac{s^2}{2} + \frac{1}{2}\left(\frac{A+C}{A-C}\right)(G-1) < \tfrac{1}{2}. \quad (112)$$

The added noise is at its minimum if $C = 0$, see Section 2.2, and then we obtain the maximum allowed amplification to be

$$G < \frac{2}{1+s^2} < 2. \quad (113)$$

Thus we find that the magic number for photon cloning, $G = 2$, sets an upper limit to the gain allowed if nonclassical features are to be retained during the amplification process. This is discussed further in Refs. [17] and [23].

Appendix: Mathematical description of quantum optical systems

A.1. *The coherent states*

The so-called coherent states were introduced into quantum optics by Glauber [24]. A useful collection of their properties and applications has recently been published [25]. Here we briefly wish to list their most useful properties for easy reference. The states are generated by the operator

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{1}{2}\alpha^*\alpha} e^{\alpha a^\dagger} e^{-\alpha^* a}. \quad (A.1)$$

These states are

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \quad (A.2)$$

The states are eigenstates of the (nonhermitean) annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle. \quad (A.3)$$

The probability of finding n photons in the state $|\alpha\rangle$ is Poissonian

$$|\langle n|\alpha\rangle|^2 = \frac{|\alpha^*\alpha|^n}{n!} e^{-\alpha^*\alpha} \quad (A.4)$$

with the average value $\alpha^*\alpha$.

The coherent states are not orthogonal but satisfy

$$|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2}. \quad (A.5)$$

Hence, if the field is in the state $|\alpha\rangle$ there is a finite probability of finding it in the state $|\alpha'\rangle$, if $|\alpha - \alpha'|$ does not greatly exceed unity. This is the quantum noise, which requires at least one photon of uncertainty even in the most coherent state.

The coherent states form an overcomplete basis set satisfying the closure relation

$$\int |\alpha\rangle \frac{d^2\alpha}{\pi} \langle\alpha| = 1. \quad (A.6)$$

The Glauber–Sudarshan representation of the density matrix provides each coherent state with a weight $P(\alpha)$ by writing

$$\varrho = \int d^2\alpha |\alpha\rangle P(\alpha) \langle\alpha|. \quad (A.7)$$

The function $P(\alpha)$ need not be positive definite everywhere and is hence no proper probability density. Smeared over any unit circle it does, however, give a proper probability because

$$\langle\alpha|\varrho|\alpha\rangle = \int d^2\beta\, e^{-|\alpha-\beta|^2} P(\beta) > 0. \quad (A.8)$$

For any normally ordered operator $M(a^\dagger, a)$ we can calculate the expectation value

$$\langle M\rangle = {\rm Tr}\, \varrho M(a^\dagger a) = \int d^2\alpha M(\alpha^*, \alpha)P(\alpha). \quad (A.9)$$

We define the second order correlation function

$$g^{(2)} = \frac{\langle a^\dagger a^\dagger aa\rangle}{|\langle a^\dagger a\rangle|^2} = \frac{\langle (a^\dagger a)^2\rangle - \langle a^\dagger a\rangle}{|\langle a^\dagger a\rangle|^2} = 1 + \frac{Q}{\langle n\rangle}. \quad (A.10)$$

Here the Mandel [26] parameter

$$Q = \frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle} - 1 \quad (A.11)$$

denotes the deviation from Poissonian statistics; if it is less than zero we have sub-Poissonian statistics. If we write $g^{(2)}$ using the P-function as

$$g^{(2)} = 1 + \int P(\alpha) \frac{(|\alpha|^2 - \langle|\alpha|^2\rangle)^2}{\langle|\alpha|^2\rangle^2} d^2\alpha, \quad (A.12)$$

we find that only for nonpositive-definite P-functions can we find sub-Poissonian statistics.

A.2. *The generating functions*

We define a generating function for normally ordered expectation values of $a^\dagger$ and a by setting

$$\chi_N(\zeta) = \mathrm{Tr}\,(\varrho e^{i\zeta a^\dagger} e^{i\zeta^* a}). \tag{A.13}$$

This gives any expectation value in the form

$$\langle a^{\dagger m} a^n \rangle = \left[\left(-i\frac{\partial}{\partial\zeta}\right)^m \left(-i\frac{\partial}{\partial\zeta^*}\right)^n \chi_N(\zeta)\right]_{\zeta=0}. \tag{A.14}$$

We take the Fourier transform

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\zeta\, e^{-i\zeta\alpha^* - i\zeta^*\alpha}\, \chi_N(\zeta), \tag{A.15}$$

and using the complex relationship

$$\frac{1}{\pi^2}\int d^2\zeta\, e^{i\zeta^*(\alpha-\beta)+i\zeta(\alpha^*-\beta^*)} = \delta^{(2)}(\alpha-\beta) \tag{A.16}$$

we write

$$\chi_N(\zeta) = \int d^2\alpha\, P(\alpha)\, e^{i\zeta\alpha^*} e^{i\zeta^*\alpha} \tag{A.17}$$

giving in (A.14) the result

$$\langle a^{\dagger m} a^n \rangle = \int d^2\alpha P(\alpha)\alpha^{*m}\alpha^n, \tag{A.18}$$

which shows that $P(\alpha)$ equals the Glauber–Sudarshan function of eq. (A.9).

The function $P(\alpha)$ does not always exist as an ordinary function because the Fourier transform (A.15) may not converge. We have namely

$$\begin{aligned}\chi_N(\zeta) &= \mathrm{Tr}\,(\varrho\, e^{i\zeta a^\dagger} e^{i\zeta^* a}) \\ &= e^{\frac{1}{2}\zeta^*\zeta}\, \mathrm{Tr}\,(\varrho\, e^{i(\zeta a^\dagger + \zeta^* a)}).\end{aligned} \tag{A.19}$$

Because this expectation value is bounded by unity the function $\chi_N(\zeta)$ may grow exponentially for large values of $|\zeta|$. This problem can be overcome, if we instead choose the characteristic function

$$\chi(\zeta) = \mathrm{Tr}\,(\varrho\, e^{i(\zeta a^\dagger + \zeta^* a)}), \tag{A.20}$$

which is the generating function for symmetrically ordered expectation values.

If we define the Fourier transform of $\chi(\zeta)$ we find after some calculations

$$\begin{aligned}W(\alpha) &= \frac{1}{\pi^2}\int d^2\zeta\, e^{-i(\alpha\zeta^*+\alpha^*\zeta)}\,\chi(\zeta) \\ &= \frac{1}{\pi}\int dy\, e^{-ivy}\, \langle u + \tfrac{1}{2}y|\varrho|\, u - \tfrac{1}{2}y\rangle\end{aligned} \tag{A.21}$$

where we have set

$$\alpha = \frac{1}{\sqrt{2}}(u + iv). \tag{A.22}$$

This proves that the function $W(\alpha)$ is the Wigner function in the u, v-representation.

We can further introduce the generating function for anti-normally ordered expectation values. It is given by

$$\begin{aligned}\chi_A(\zeta) &= \mathrm{Tr}\,(\varrho\, e^{i\zeta^* a} e^{i\zeta a^\dagger}) \\ &= e^{-\frac{1}{2}\zeta^*\zeta}\,\chi(\zeta) = e^{-\zeta^*\zeta}\,\chi_N(\zeta).\end{aligned} \tag{A.23}$$

Its Fourier transform is found to be

$$\begin{aligned}Q(\alpha) &= \frac{1}{\pi^2}\int d^2\zeta\, \chi_N(\zeta)\, e^{-i\zeta\alpha^* - i\zeta^*\alpha} \\ &= \frac{1}{\pi}\langle\alpha|\varrho|\alpha\rangle.\end{aligned} \tag{A.24}$$

In addition to guaranteed existence, just like the function $W(\alpha)$, this has the additional property of being positive definite. It hence corresponds to a genuine probability density, as found in relation (A8).

A3. *The master equations*

If we want to derive the equations of motion for the various distribution functions, it is useful to start from the normally ordered χ_N. To this end we must bring the terms of the master equation (45) into normal form and calculate the evolution of the expectation value of $\langle e^{i\beta a^\dagger} e^{i\beta^* a}\rangle$. For this we use the relations

$$aa^\dagger = a^\dagger a + 1 \tag{A.30}$$

and

$$e^{i\beta a} a^\dagger = a^\dagger e^{i\beta a} + i\beta e^{i\beta a}. \tag{A.31}$$

We find after some algebra

$$\frac{\partial\chi_N}{\partial t} = \frac{A-C}{2}\left(\beta\frac{\partial\chi_N}{\partial\beta} + \beta^*\frac{\partial\chi_N}{\partial\beta^*}\right) - A\beta^*\beta\chi_N. \tag{A.32}$$

Applying the Fourier transform (A.15) gives the transformations

$$\beta \to i\frac{\partial}{\partial\alpha^*} \qquad \frac{\partial}{\partial\beta} \to i\alpha^*, \tag{A.33}$$

which applied to eq. (A.32) give directly the terms of eq. (57).

If we wish to derive the master equation for the Wigner function we write

$$\chi_N(\beta) = e^{\frac{1}{2}\beta^*\beta}\,\chi(\beta) \tag{A.34}$$

and calculate

$$\left(\beta\frac{\partial}{\partial\beta} + \beta^*\frac{\partial}{\partial\beta^*}\right)\chi_N(\beta) = e^{\frac{1}{2}\beta^*\beta}\left[\beta^*\beta\chi + \beta\frac{\partial\chi}{\partial\beta} + \beta^*\frac{\partial\chi}{\partial\beta^*}\right]. \tag{A.35}$$

When these relations are inserted into eq.(A.32) we find

$$\frac{\partial\chi}{\partial t} = \frac{A-C}{2}\left(\beta\frac{\partial\chi}{\partial\beta} + \beta^*\frac{\partial\chi}{\partial\beta^*}\right) - \frac{A+C}{2}\beta^*\beta\chi. \tag{A.36}$$

Fourier transforming we find

$$\begin{aligned}\frac{\partial W(\alpha)}{\partial t} &= -\frac{A-C}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)W(\alpha) \\ &\quad + \frac{A+C}{2}\frac{\partial^2 W(\alpha)}{\partial\alpha\partial\alpha^*}.\end{aligned} \tag{A.37}$$

Using the relation (A.22) we find this to agree with the result (106).

For the anti-normally ordered characteristic function

$$\chi_A(\beta) = e^{-\frac{1}{2}\beta^*\beta}\,\chi(\beta) \tag{A.38}$$

we find the relation

$$\frac{\partial\chi_A}{\partial t} = \frac{A-C}{2}\left(\beta\frac{\partial\chi_A}{\partial\beta} + \beta^*\frac{\partial\chi_A}{\partial\beta^*}\right) - C\beta^*\beta\chi_A. \tag{A.39}$$

A Fourier transform shows that χ_A has its diffusion coefficient determined entirely by the damping coefficient C.

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