G. BAYM

$$\langle\Phi|\psi^\dagger(\mathbf{r})\psi(\mathbf{r})|\Phi\rangle = \frac{1}{V}\sum_{\mathbf{p}} n_{\mathbf{p}} \equiv n. \tag{19-79}$$

The calculation of the pair correlation function begins with Eq. (19-71), whose form is equally valid for bosons. The expectation value $\langle\Phi|a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{q}'} a_{\mathbf{p}'}|\Phi\rangle$ is nonvanishing only if $\mathbf{p} = \mathbf{p}'$, $\mathbf{q} = \mathbf{q}'$ or $\mathbf{p} = \mathbf{q}'$, $\mathbf{q} = \mathbf{p}'$. These are not distinct cases though if $\mathbf{p} = \mathbf{q}$. Thus we have

$$\begin{aligned}\langle\Phi|a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{q}'} a_{\mathbf{p}'}|\Phi\rangle &= (1-\delta_{\mathbf{pq}})(\delta_{\mathbf{pp}'}\delta_{\mathbf{qq}'}\langle\Phi|a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}|\Phi\rangle \\ &\quad + \delta_{\mathbf{pq}'}\delta_{\mathbf{qp}'}\langle\Phi|a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}}|\Phi\rangle) + \delta_{\mathbf{pq}}\delta_{\mathbf{pp}'}\delta_{\mathbf{qq}'}\langle\Phi|a_{\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger a_{\mathbf{p}} a_{\mathbf{p}}|\Phi\rangle \\ &= (1-\delta_{\mathbf{pq}})(\delta_{\mathbf{pp}'}\delta_{\mathbf{qq}'} + \delta_{\mathbf{pq}'}\delta_{\mathbf{qp}'})n_{\mathbf{p}}n_{\mathbf{q}} + \delta_{\mathbf{pq}}\delta_{\mathbf{pp}'}\delta_{\mathbf{qq}'}n_{\mathbf{p}}(n_{\mathbf{p}}-1).\end{aligned} \tag{19-80}$$

Putting this into (19-71) we find

$$\begin{aligned}&\langle\Phi|\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})|\Phi\rangle \\ &= n^2 + \left|\frac{1}{V}\sum_{\mathbf{p}} n_{\mathbf{p}} e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\right|^2 - \frac{1}{V^2}\sum_{\mathbf{p}} n_{\mathbf{p}}(n_{\mathbf{p}}+1).\end{aligned} \tag{19-81}$$

This result differs from the fermion result in two respects: the sign of the second term is positive (a consequence of the exchange symmetry of boson wave functions), and the presence of the last term, which arises because one can have many bosons in the same state.

For example, if *all* the particles are in only one state $\mathbf{p}_0$, then (19-81) becomes

$$n^2 + n^2 - \left[\frac{1}{V^2}N(N+1)\right] = \frac{N(N-1)}{V^2}. \tag{19-82}$$

This says simply that the relative amplitude for removing the first particle is N/V, while the amplitude for removing the second is (N − 1)/V, since there are only N − 1 particles left after removing the first.

Consider next the case that $n_{\mathbf{p}}$ is a smoothly varying distribution. To be definite let us take

$$n_{\mathbf{p}} = c e^{-\alpha(\mathbf{p}-\mathbf{p}_0)^2/2}, \tag{19-83}$$

which essentially represents a beam of particles of momentum centered, with a Gaussian spread, about $\mathbf{p}_0$. If we take the limit of

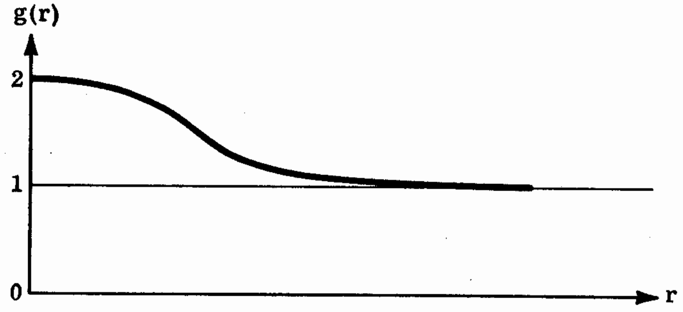

Fig. 19-3
The pair correlation function for noninteracting spin zero bosons.

large volume, keeping n fixed, then the last term in (19-81) is of the order 1/V smaller than the first two terms, and we can drop it. Converting the sums to integrals, (19-81) becomes

$$\begin{aligned}\langle\Phi|\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r})|\Phi\rangle &\equiv n^2 g(\mathbf{r}-\mathbf{r}') \\ &= n^2 + \left|\int\frac{d^3p}{(2\pi)^3} n_{\mathbf{p}} e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\right|^2 = n^2\left(1 + e^{-(\mathbf{r}-\mathbf{r}')^2/\alpha}\right).\end{aligned} \tag{19-84}$$

The $e^{-(\mathbf{r}-\mathbf{r}')^2/\alpha}$ term is the effect of exchange. We see that it *increases* the probability for two bosons to be found at small separations. In fact, the probability for finding two bosons right on top of each other, $\mathbf{r} = \mathbf{r}'$, is *twice* the value for finding two at a large $|\mathbf{r} - \mathbf{r}'|$, as in Fig. 19-3.

THE HANBURY-BROWN AND TWISS EXPERIMENT

The *Hanbury-Brown and Twiss experiment* [1] provides a simple way of observing this tendency of bosons to clump together. Basically, the experiment measures the probability of observing two photons simultaneously at different points in a beam of incoherent light (which as we've seen, can be described in terms of the occupation numbers of the photon states). The actual measuring apparatus uses a half silvered mirror, Fig. 19-4, to split the beam into two identical beams; this avoids the problem of one detector

[1] *Nature* 177, 27 (1956); 178, 1447 (1956).

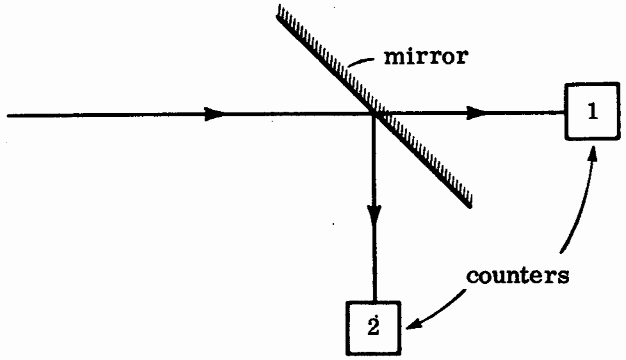

Fig. 19-4
The half-silvered mirror and counters in the Hanbury-Brown and Twiss experiment.

casting a shadow on the other. The amplitude for a photon to be transmitted, or reflected by the mirror, is $1/\sqrt{2}$. Hanbury-Brown and Twiss measured the light intensities $I_1(t)$ observed in detector 1 at time t, and $I_2(t+\tau)$ observed in detector 2 at a later time $t+\tau$, and averaged the product of the intensities over t, keeping τ fixed. This is equivalent to determining the relative probability of observing two photons at two points separated by a distance $c\tau$ in the beam, where c is the speed of light. The observed average correlated intensities $\overline{I_1(t)I_2(t+\tau)}$, as a function of τ, turned out to have just the form we derived for g(r), in Fig. 19-3, with $r = c\tau$.

This experiment looks like a fine verification of the laws of quantum mechanics for identical bosons. On the contrary, it can be understood completely in terms of classical electromagnetism. What the experiment teaches us is that the boson nature of the photon is already contained in the superposition principle obeyed by classical electromagnetic fields. To see this, let us suppose, as in Fig. 19-5, that in the source of the beam there are just two emitters, A and B. Assume that A emits coherent light with amplitude α and wave num-

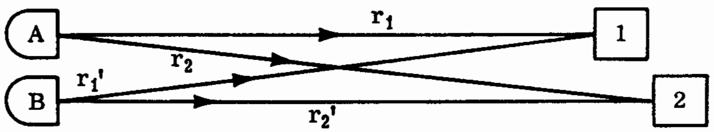

Fig. 19-5

ber k, B emits coherent light with amplitude β and wave number k', that the relative phase of these two sources is random, and that the light from each has the same polarization. The light from A falling on detector 1 has amplitude αe^{ikr_1} where r_1 is the distance to detector 1 from A; the light from B on 1 has amplitude $\beta e^{ik'r_1'}$ where r_1' is the distance from B to 1. Thus the total amplitude falling on 1, according to the superposition principle, is

$$a_1 = \alpha e^{ikr_1} + \beta e^{ik'r_1'} \tag{19-85}$$

(times some polarization vector) while the intensity is

$$I_1 = |\alpha|^2 + |\beta|^2 + 2\ \mathrm{Re}\ \alpha^*\beta^{i(k'r_1' - kr_1)}. \tag{19-86}$$

If we average over the relative phase of α and β (equivalent to averaging over t in the Hanbury-Brown and Twiss experiment) we find

$$\bar{I}_1 = |\alpha|^2 + |\beta|^2. \tag{19-87}$$

Similarly the amplitude falling on the second counter is

$$a_2 = \alpha e^{ikr_2} + \beta e^{ik'r_2'} \tag{19-88}$$

(times a polarization vector) where r_2 is the distance from A to 2 and r_2' is the distance from B to 2. Thus

$$I_2 = |\alpha|^2 + |\beta|^2 + 2\ \mathrm{Re}\ \alpha^*\beta\ e^{i(k'r_2' - kr_2)} \tag{19-89}$$

and averaged,

$$\bar{I}_2 = |\alpha|^2 + |\beta|^2. \tag{19-90}$$

The product of the averaged intensities, $\bar{I}_1\bar{I}_2$ is independent of the distance between detectors 1 and 2. However, the product of the intensities is

$$I_1I_2 = |a_1a_2|^2 = |\alpha^2 e^{ik(r_1+r_2)} + \beta^2 e^{ik'(r_1'+r_2')} + \alpha\beta(e^{ikr_1}e^{ik'r_2'} + e^{ik'r_1'}e^{ikr_2})|^2; \tag{19-91}$$

multiplying this out and averaging over the relative phase of α and β (which eliminates the terms proportional to $\alpha\beta|\alpha|^2$, $\alpha\beta|\beta|^2$, etc.) we find

$$\overline{I_1I_2} = |\alpha|^4 + |\beta|^4 + |\alpha|^2|\beta|^2|e^{ikr_1}e^{ik'r_2'} + e^{ik'r_1'}e^{ikr_2}|^2$$
$$= \bar{I}_1\bar{I}_2 + 2|\alpha|^2|\beta|^2 \cos[k'(r_1' - r_2') - k(r_1 - r_2)]. \tag{19-92}$$

For a well collimated beam $r_1 - r_2 \approx r_1' - r_2'$ so that (19-93) becomes

$$\overline{I_1I_2} = \bar{I}_1\bar{I}_2 + 2|\alpha|^2|\beta|^2 \cos[(k' - k)(r_1 - r_2)]. \tag{19-93}$$

Thus we find a term in the correlated intensities that depends on the relative separation of the two detectors; this term is maximum when the two detectors are at the same point. Now finally we should average the result (19-93) over all the different k and k' present in the beam. Then we find, for a Gaussian distribution, exactly the form (19-84). The photon bunching effect is thus a consequence of the superposition principle for light applied to noisy sources.[2]

From a quantum mechanical point of view we can interpret the three terms on the right side (19-91) as follows. The α^2 term is the amplitude for the two observed photons both to have come from A; this leads to the $|\alpha|^4$ term in (19-92). The β^2 term is the amplitude for them both to have come from B; this produces the $|\beta|^4$ term in (19-92). The $\alpha\beta$ term is the amplitude for one of the photons to have come from A and the other from B. There are two ways for this to occur — the photon from A can strike 1 while the photon from B strikes 2, or vice versa. These two ways are indistinguishable, and it is just the interference between them that leads to the cos term in (19-92).

Try to imagine the results of the Hanbury-Brown and Twiss experiment if it were performed with a beam of electrons.

THE HAMILTONIAN

There is still one very important operator we have not yet learned how to write in second quantized language — the Hamiltonian. If the particles interact by means of a two-body potential $v(\mathbf{r} - \mathbf{r}')$ then the interaction energy operator is

$$\mathcal{V} = \frac{1}{2}\sum_{ss'}\int d^3r\, d^3r'\, v(\mathbf{r} - \mathbf{r}')\psi_s^\dagger(\mathbf{r})\psi_{s'}^\dagger(\mathbf{r}')\psi_{s'}(\mathbf{r}')\psi_s(\mathbf{r}). \tag{19-94}$$

[2] This is discussed further by E. Purcell in *Nature* **178**, 1449 (1956).

Note carefully the order of the operators. The order is the same as that used in (19-70) to determine the pair distribution function. It is left as an exercise to verify (19-94) by writing out a matrix element $\langle\Phi'|\mathcal{V}|\Phi\rangle$ of (19-94) in terms of the wave functions of the states. We can interpret the potential energy operator (19-94) as first trying to remove particles from $\mathbf{r}$ and $\mathbf{r}'$; if it is successful it counts a $v(\mathbf{r} - \mathbf{r}')$ and then replaces the particles, replacing the last removed particle first. It then sums over all *pairs* of points $\mathbf{r}$ and $\mathbf{r}'$, whence the factor $1/2$.

The second quantized Hamiltonian for particles of mass m acting pairwise is, using (19-58), thus

$$H = \sum_s \int d^3r \frac{1}{2m} \nabla\psi_s^\dagger(\mathbf{r}) \cdot \nabla\psi_s(\mathbf{r})$$
$$+ \frac{1}{2}\sum_{ss'}\int d^3r\, d^3r'\, v(\mathbf{r} - \mathbf{r}')\psi_s^\dagger(\mathbf{r})\psi_{s'}^\dagger(\mathbf{r}')\psi_{s'}(\mathbf{r}')\psi_s(\mathbf{r}). \tag{19-95}$$

Let us evaluate the ground state energy of a gas of spin $1/2$ fermions, treating the interaction v as a perturbation. To lowest order the energy is simply the kinetic energy,

$$E^{(0)} = \sum_{ps} \frac{p^2}{2m} = 2\int_0^{p_f} V \frac{d^3p}{(2\pi)^3}\frac{p^2}{2m} = \frac{3}{5}\frac{p_f^2}{2m} N. \tag{19-96}$$

The average kinetic energy per particle is $3/5$ of the Fermi energy. The first-order change $E^{(1)}$ in the energy is simply the expectation value of $\mathcal{V}$ in the unperturbed ground state. Thus

$$E^{(1)} = \frac{1}{2}\int d^3r\, d^3r'\, v(\mathbf{r} - \mathbf{r}') \sum_{ss'} \langle\Phi_0|\psi_s^\dagger(\mathbf{r})\psi_{s'}^\dagger(\mathbf{r}')\psi_{s'}(\mathbf{r}')\psi_s(\mathbf{r})|\Phi_0\rangle$$
$$= \frac{1}{2}\int d^3r\, d^3r'\, v(\mathbf{r} - \mathbf{r}') \sum_{ss'} \left(\frac{n}{2}\right)^2 g_{ss'}(\mathbf{r} - \mathbf{r}'), \tag{19-97}$$

where $g_{ss'}(\mathbf{r} - \mathbf{r}')$ is the pair correlation function. Using (19-73), and (19-75), we find

$$E^{(1)} = \frac{1}{2}\int d^3r\, d^3r'\, v(\mathbf{r} - \mathbf{r}')[n^2 - \sum_s G_s(\mathbf{r} - \mathbf{r}')^2]. \tag{19-98}$$

The n^2 term gives $Nnv_0/2$, where $v_0 = \int d^3r\, v(\mathbf{r})$; it represents the average interaction of a uniform density of particles with itself, leaving out all correlation effects. This energy is called the direct, or Hartree, energy. The second term, called the *exchange energy*,