Classical Capacity of the Lossy Bosonic Channel: The Exact Solution

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The classical capacity of the lossy bosonic channel is calculated exactly. It is shown that its Holevo information is not superadditive, and that a coherent-state encoding achieves capacity. The capacity of far-field, free-space optical communications is given as an example.

A principal goal of quantum information theory is evaluating the information capacities of important communication channels. At present — despite the many efforts that have been devoted to this endeavor and the theoretical advances they have produced [1] — exact capacity results are known for only a handful of channels. In this Letter we consider the lossy bosonic channel, and we develop an exact result for its classical capacity C, i.e., the number of bits that it can communicate reliably per channel use. The lossy bosonic channel consists of a collection of bosonic modes that lose energy en route from the transmitter to the receiver. Typical examples are free-space or optical fiber transmission, in which photons are employed to convey the information. The classical capacity of the lossless bosonic channel — whose transmitted states arrive undisturbed at the receiver — was derived in [2,3]. When there is loss, however, the received state is in general different from the transmitted state, and quantum mechanics requires that there be an accompanying quantum noise source. In [4] a first step toward the capacity of such channels was given by considering only separable encoding procedures. Here, on the contrary, it is proven that the optimal encoding is indeed separable. We obtain the value of C in the presence of loss when the quantum noise source is in the vacuum state, i.e., when it injects the minimum amount of noise into the receiver. Our derivation proceeds by developing an upper bound for C and then showing that this bound coincides with the lower bound on C reported in [5,6]. Our upper bound results from comparing the capacity of the lossy channel to that of the lossless channel whose average input energy matches the average output energy constraint for the lossy case [7]. This argument is analogous to the derivation of the classical capacity of the erasure channel [8]. The lower bound comes from calculating the Holevo information for appropriately coded coherent-state inputs. Thus, because the two bounds coincide, we not only have the capacity of the lossy bosonic channel, but we also know that capacity can be achieved by transmitting coherent states.

Classical capacity.—The classical capacity of a channel can be expressed in terms of the Holevo information

$$\chi(p_j, \sigma_j) \equiv S\left(\sum_j p_j \sigma_j\right) - \sum_j p_j S(\sigma_j), \qquad (1)$$

where p_j are probabilities, σ_j are density operators, and $S(\varrho) \equiv -\mathrm{Tr}[\varrho \log_2 \varrho]$ is the von Neumann entropy. Since it is not known if χ is additive, C must be calculated by maximizing the Holevo information over successive uses of the channel, so that $C = \sup_n(C_n/n)$ with

$$C_n = \max_{p_j, \sigma_j} \chi(p_j, \mathcal{N}^{\otimes n}[\sigma_j]), \qquad (2)$$

where the states σ_j live in the Hilbert space $\mathcal{H}^{\otimes n}$ of n successive uses of the channel and $\mathcal{N}$ is the completely positive map that describes the channel [9]. In our case, $\mathcal{H}$ is the Hilbert space associated with the bosonic modes used in the communication and $\mathcal{N}$ is the loss map. Because $\mathcal{H}$ is infinite dimensional, C_n diverges unless the maximization in Eq. (2) is constrained: here we assume that the mean energy of the input state in each of the n realizations of the channel is a fixed quantity $\mathcal{E}$. For multimode bosonic channels, $\mathcal{N}$ is given by $\bigotimes_k \mathcal{N}_k$, where $\mathcal{N}_k$ is the loss map for the kth mode, which can be obtained, tracing away the vacuum noise mode b_k, from the Heisenberg evolution

$$a'_k = \sqrt{\eta_k}\, a_k + \sqrt{1 - \eta_k}\, b_k, \qquad (3)$$

with a_k and a'_k being the annihilation operators of the input and output modes and $0 \leq \eta_k \leq 1$ is the mode transmissivity (quantum efficiency).

The main result of this Letter is that the capacity of the lossy bosonic channel, in bits per channel use, is

$$C = \max_{N_k} \sum_k g(\eta_k N_k), \qquad (4)$$

where $g(x) \equiv (x + 1)\log_2(x + 1) - x\log_2 x$ and where the maximization is performed on the modal average photon-number sets $\{N_k\}$ that satisfy the energy constraint

$$\sum_k \hbar \omega_k N_k = \mathcal{E}, \qquad (5)$$

(ω_k is the frequency of the kth mode).

We derive Eq. (4) by giving coincident lower and upper bounds for C. The right-hand side of Eq. (4) was shown, in [6], to be a lower bound for C by generalizing the narrowband analysis of [5]. This expression was obtained from Eq. (2) by calculating χ for $n = 1$ under the following encoding: in every mode k we use a mixture of coherent states $|\mu\rangle_k$ weighted with the Gaussian probability distribution

$$p_k(\mu) = \exp[-|\mu|^2/N_k]/(\pi N_k). \qquad (6)$$

This corresponds to feeding the channel the input state

$$\varrho = \bigotimes_k \int d\mu \, p_k(\mu)|\mu\rangle_k\langle\mu|, \qquad (7)$$

which is a thermal state that contains no entanglement or squeezing. The right-hand side of Eq. (4) is also an upper bound for C. To see that this is so, let $\bar{p}_j$, $\bar{\sigma}_j$ be the optimal encoding on n uses of the channel, which gives the capacity C_n of Eq. (2). The definition of χ and the subadditivity of the von Neumann entropy allow us to write

$$C_n \leq S(\mathcal{N}^{\otimes n}[\bar{\sigma}]) \leq \sum_{l=1}^{n} \sum_k S(\mathcal{N}_k[\varrho_k^{(l)}]), \qquad (8)$$

where $\bar{\sigma} \equiv \sum_j \bar{p}_j \bar{\sigma}_j$ and $\mathcal{N}_k[\varrho_k^{(l)}]$ is the reduced density operator of the kth mode in the lth realization of the channel, which is obtained from $\mathcal{N}^{\otimes n}[\bar{\sigma}]$ by tracing over all the other modes and over the other $n - 1$ channel realizations. The first inequality in Eq. (8) comes from bounding C_n by the amount of information that can be transmitted through a lossless channel with input state $\mathcal{N}^{\otimes n}[\bar{\sigma}]$, viz., the output of the lossy channel with optimal input state $\bar{\sigma}$ [7]. Now let $N_k^{(l)}$ be the average photon number for the state $\varrho_k^{(l)}$; the set $\{N_k^{(l)}\}$ must satisfy the energy constraint (5) for all l [10]. Moreover, the loss will leave only $\eta_k N_k^{(l)}$ photons, on average, in the corresponding output state $\mathcal{N}_k[\varrho_k^{(l)}]$. This implies that

$$S(\mathcal{N}_k[\varrho_k^{(l)}]) \leq g(\eta_k N_k^{(l)}), \qquad (9)$$

where the inequality follows from the fact that the term on the right is the maximum entropy associated with states that have $\eta_k N_k^{(l)}$ photons on average [3,11]. Introducing Eq. (9) into (8), we obtain the desired result

$$C_n \leq \sum_{l=1}^{n} \sum_k g(\eta_k N_k^{(l)}) \leq n \max_{N_k} \sum_k g(\eta_k N_k), \qquad (10)$$

where the maximization is performed over the sets $\{N_k\}$ that satisfy Eq. (5). Because Eq. (10) holds for any n, we conclude that the right-hand side of (4) is indeed also an upper bound for C.

Discussion.—Some important consequences derive from our analysis. First, capacity is achieved by a single use of the channel ($n = 1$) employing random coding—factorized over the channel modes—on coherent states as shown in Eq. (7). This means that, at least for this channel, entangled code words are not necessary and that the Holevo information is not superadditive. Notice that the lossy bosonic channel can accommodate entanglement among successive uses of the channel, as well as entanglement among different modes in each channel use. Surprisingly, neither of these two strategies is necessary to achieve capacity. Nor is it necessary to use any nonclassical state, such as a photon-number state or a squeezed state, to achieve capacity; classical (coherent-state) light is all that is needed. Classical light suffices because the loss map $\mathcal{N}$ simply contracts coherent-state code words in phase space toward the vacuum state. Coherent states retain their purity in this process, and hence the nonpositive part of the Holevo information—the second term of the right-hand side of Eq. (1)—retains its maximum value of zero. Despite the preceding properties, quantum effects are relevant to communication over the lossy bosonic channel. For example, our proof does not exclude the possibility of achieving capacity using quantum encodings, and such encodings may have lower error probabilities, for finite-length block codes, than those of the capacity-achieving coherent-state encoding. This is certainly true for the lossless case. In particular, it was already known that C can be achieved with a number-state alphabet [2,3]; our work shows that there is also a coherent-state encoding that achieves capacity for this case. [The two procedures employ the same average input state, Eq. (7)]. However, the probability of the receiver confusing any two distinct finite-length number-state code words is zero in the lossless case, whereas it is positive for all pairs of finite-length coherent-state code words. The lossless case also provides an example of the possible role of quantum effects at the receiver: the optimal coherent-state system uses a classical transmitter, but its detection strategy, can be highly nonclassical [9]. In contrast, the optimal number-state system for the lossless channel requires a nonclassical light source, but its receiver uses simple modal photon counting.

How well can we approach this capacity using conventional decoding procedures? Using the coherent-state encoding of Eq. (7) with either heterodyne or homodyne detection, the amount of information that can be reliably transmitted is

$$I = \max_{N_k} \sum_k \xi \log_2(1 + \eta_k N_k/\xi^2), \qquad (11)$$

where $\xi = 1/2$ for homodyne and $\xi = 1$ for heterodyne, and where, as usual, the maximization must be performed under the energy constraint (5). Equation (11) has been obtained by summing over k the Shannon capacities for the appropriate detection procedure [3]. In

general $I < C$: heterodyne or homodyne detection cannot be used to achieve the capacity. However, heterodyne is asymptotically optimal in the limit of large numbers of photons in all modes, $N_k \to \infty$ for all k, because $g(x)/\log_2(x) \to 1$ as $x \to \infty$.

The capacity expression C can be simplified by using standard variational techniques to perform the constrained maximization in Eq. (4), yielding [6]

$$C = \sum_k g(\eta_k N_k(\beta)), \qquad (12)$$

where $N_k(\beta)$ is the optimal photon-number distribution

$$N_k(\beta) = \frac{1/\eta_k}{e^{\beta\hbar\omega_k/\eta_k} - 1}, \qquad (13)$$

with β being a Lagrange multiplier that is determined through the constraint on average transmitted energy.

In the following sections we calculate the capacities of some bosonic channels. The first two examples help clarify the derivation of Eq. (4); the last is a realistic model of frequency-dependent lossy communication, on which we also evaluate the performance of homodyne and heterodyne detection.

Narrowband channel.—Consider the narrowband channel in which a single mode of frequency ω is employed. In this case, Eq. (12) becomes

$$C = g\left(\frac{\eta\mathcal{E}}{\hbar\omega}\right), \qquad (14)$$

where $N = \mathcal{E}/(\hbar\omega)$ is the average photon number at the input. Equation (14) was conjectured in [5], where it was given as a lower bound on C. The following simple argument shows that $g(\eta N)$ is also an upper bound for C. Consider the lossless channel that employs ηN photons on average per channel use. Its capacity is given by $\max_\varrho S(\varrho)$, where the maximization is performed over input states ϱ with mean energy $\mathcal{E}' = \eta\hbar\omega N$ [12]. The maximum, computed through variational techniques, is $g(\eta N)$ [3,11]. The lossless channel cannot have a lower capacity than the lossy channel, because both have the same average received energy, and the set of receiver density operators achievable over the lossy channel is a proper subset of those achievable in the lossless system [7]. This implies that $g(\eta N)$ is an also upper bound on C and hence equal to C.

Frequency-independent loss.—Now consider a broadband channel with uniform transmissivity, $\eta_k = \eta$, that employs a set of frequencies $\omega_k = k\,\delta\omega$ for $k \in \mathbb{N}$. In this case, Eq. (12) gives [13]

$$C = \frac{\sqrt{\eta}}{\ln 2}\sqrt{\frac{\pi\mathcal{P}}{3\hbar}}\,\mathcal{T}, \qquad (15)$$

where $\mathcal{T} = 2\pi/\delta\omega$ is the transmission time, and $\mathcal{P} = \mathcal{E}/\mathcal{T}$ is the average transmitted power. Equation (15) was derived for the lossless case ($\eta = 1$) in [2] and was shown to provide a lower bound on C in [6]. In order to show that

the right-hand side of Eq. (15) is also an upper bound, consider the lossless broadband channel in which the average *input power is equal to* $\eta\mathcal{P}$, viz., the average output power of the lossy channel. According to [2], the capacity of this channel is $(\sqrt{\pi\eta\mathcal{P}/3})\mathcal{T}/\ln 2$, which coincides with the right-hand side of Eq. (15). The reasoning given above for the single-mode case now implies that the broadband lossless channel's capacity cannot be less than that of the broadband lossy channel, thus completing the proof.

Far-field, free-space optical communication.—Consider the free-space optical communication channel in which the transmitter and the receiver communicate through circular apertures of areas A_t and A_r that are separated by an L-m–long propagation path. At frequency ω there will only be a single spatial mode in the transmitter aperture that couples appreciable power to the receiver aperture when the Fresnel number $D(\omega) \equiv A_t A_r(\omega/2\pi cL)^2$ satisfies $D(\omega) \ll 1$ [14]. This is the far-field power transfer regime at frequency ω, and $D(\omega)$ is the transmissivity achieved by the optimal spatial mode. A broadband far-field channel results when the transmitter and receiver use the optimal spatial modes at frequencies up to a critical frequency ω_c, with $D(\omega_c) \ll 1$. In this case we use $\eta_k = D(\omega_k)$ in Eq. (12), and the capacity C becomes [13]

$$C = \frac{\omega_c \mathcal{T}}{2\pi y_0}\int_0^{y_0} dx\, g\left(\frac{1}{e^{1/x} - 1}\right), \qquad (16)$$

where y_0 is a dimensionless parameter inversely proportional to the Lagrange multiplier β, which is determined
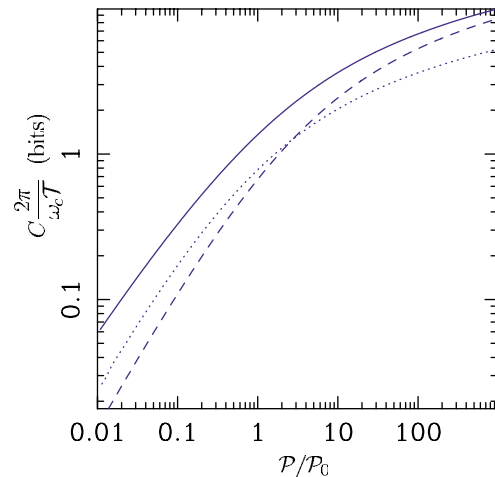

FIG. 1 (color online). Capacities of the far-field free-space optical channel as a function of the input power $\mathcal{P}$ [in the plot $\mathcal{P}_0 \equiv 2\pi\hbar c^2 L^2/(A_t A_r)$]. The solid curve is the capacity C from Eq. (16), the other two curves are the capacities I from Eq. (18) achievable with coherent states and heterodyne detection (dashed curve) or coherent states and homodyne detection (dotted curve). Note that the heterodyne detection I approaches the optimal capacity C in the high-power limit.
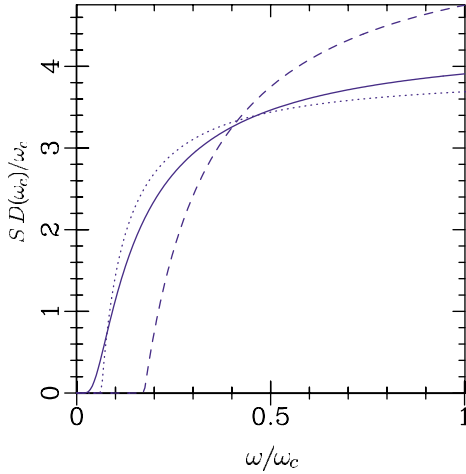
FIG. 2 (color online). Power spectrum $S \equiv \omega_k N_k$ for the far-field free-space channel plotted versus frequency in the continuum regime [13]. The solid curve is for optimal capacity, the dotted curve is for homodyne detection, and the dashed curve is for heterodyne detection. Here $\mathcal{P}/\mathcal{P}_0 = 3$. In contrast to the frequency-independent lossy channel, all of these coherent-state encodings preferentially employ high frequencies instead of low frequencies. This marked change in spectral shaping is due to the transmissivity's having a quadratic dependence on ω.

from the power constraint

$$\mathcal{P} = \frac{2\pi\hbar c^2 L^2}{A_t A_r} \int_0^{y_0} \frac{dx}{x} \frac{1}{e^{1/x} - 1}. \tag{17}$$

Although C is proportional to the maximum frequency ω_c, this factor cannot be increased without bound, for fixed transmitter and receiver apertures, because of the far-field assumption. Figure 1 plots C versus $\mathcal{P}$ obtained from numerical evaluation of Eqs. (16) and (17).

To compare the capacity of Eq. (16) with the information transmitted using heterodyne or homodyne detection, we perform the Eq. (11) maximization. The Lagrange multiplier technique gives the optimal value $N_k(\beta) = \max\{1/(\beta\hbar\omega_k) - \xi^2/\eta_k , 0\}$, plotted in Fig. 2. [Notice that the non-negativity of this solution forbids the use of frequencies lower than $\omega_0 \equiv \xi^2 \beta\hbar\omega_c^2/D(\omega_c)$.] With this photon-number distribution, Eq. (11) becomes

$$I = \xi\omega_c \mathcal{T}(1/y_0 - 1 + \ln y_0)/(2\pi \ln 2), \tag{18}$$

where y_0 is now determined from the condition $\mathcal{P} = \xi^2 2\pi\hbar c^2 L^2 (y_0 - 1 - \ln y_0)/(A_r A_s)$. We have plotted I versus P in Fig. 1 for heterodyne and homodyne detection. At low power, the noise advantage of homodyne makes its capacity higher than that of heterodyne. At high-power levels heterodyne prevails thanks to its bandwidth advantage, and its capacity approaches C asymptotically.

In conclusion, we have derived the classical capacity of the lossy multimode bosonic channel when the average energy devoted to the transmission is bounded.

Interestingly, quantum features of the signals (such as entanglement or squeezing) are not required to achieve capacity, because an optimal coherent-state encoding exists. At the decoding stage, however, quantum effects might still be necessary (e.g., in the form of joint measurements on the output) as standard homodyne and heterodyne measurements are not optimal, except for the high-photon-number regime where heterodyne detection is asymptotically optimal. The focus of this Letter has been the lossy channel with minimal (vacuum state) noise. A more general treatment would include non-vacuum noise, and would allow for amplification.

[1] C. H. Bennett and P.W. Shor, IEEE Trans. Inf. Theory **44**, 2724 (1998); A. S. Holevo, Tamagawa Univ. Res. Rev. 4 (1998), quant-ph/9809023; M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000), and references therein.

[2] H. P. Yuen and M. Ozawa, Phys. Rev. Lett. **70**, 363 (1992).

[3] C. M. Caves and P. D. Drummond, Rev. Mod. Phys. **66**, 481 (1994), and references therein.

[4] A. S. Holevo, M. Sohma, and O. Hirota, Phys. Rev. A **59**, 1820 (1999); M. Sohma and O. Hirota, *Recent Research and Developments in Optics* (Research Signpost, Trivandrum, India, 2001), Vol. 1, p. 146.

[5] A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001).

[6] V. Giovannetti, S. Lloyd, L. Maccone, and P.W. Shor, Phys. Rev. Lett. **91**, 047901 (2003); Phys. Rev. A **68**, 062323 (2003).

[7] H. P. Yuen, in *Quantum Squeezing*, edited by P. D. Drummond and Z. Spicek (Springer Verlag, Berlin, 2003).

[8] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, Phys. Rev. Lett. **78**, 3217 (1997).

[9] A. S. Holevo, IEEE Trans. Inf. Theory **44**, 269 (1998); P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters, Phys. Rev. A **54**, 1869 (1996); B. Schumacher and M. D. Westmoreland, Phys. Rev. A **56**, 131 (1997).

[10] The concavity of $g(x)$ implies that this constraint is not strictly necessary: it suffices to require that the average over l be fixed, i.e. $\sum_{k,l}\hbar\omega_k N_k^{(l)}/n = \mathcal{E}$.

[11] J. D. Bekenstein, Phys. Rev. D **23**, 287 (1981); Phys. Rev. A **37**, 3437 (1988).

[12] In the noiseless case the maximization of the Holevo quantity (2) yields the von Neumann entropy of the input state, which is a subadditive quantity.

[13] Notice that in the high-power regime, the sums in Eqs. (5) and (12) can be replaced with integrals.

[14] D. Slepian, J. Opt. Soc. Am. **55**, 1110 (1965); H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory **24**, 657 (1978).