Inductance Effects in the Persistent Current Qubit

Donald S. Crankshaw and Terry P. Orlando

Abstract—**A general method is illustrated to show that the Hamiltonian for circuits of Josephson junctions can be expanded in terms of three Hamiltonians: a Hamiltonian representative of the inductance-free circuit, a Hamiltonian in the form of an harmonic oscillator for the inductance effects of the circulating currents, and a small correction term. This method is used to show that the inductive effects are a small correction to the difference in energy levels in the persistent current qubit.**

Index Terms—**inductance, quantum computation, qubit, SQUID**

I. Introduction

Dissipationless, superconducting circuits have been proposed as qubits for quantum computation. The two logic states of the qubits are chosen as two convenient energy levels of the quantum mechanical circuit. Usually these states are either described by charge states differing by a superelectron [1,2] or by flux states which differ by the amount of flux stored in a closed loop [3]. Moreover, two main types of flux-based qubits have been proposed. The first type, the rf SQUID qubit, is a single loop with one Josephson junction. The inductance of the loop L generates states of nearly a flux quantum, Φ_0. The circulating current is of the order of the critical current of a junction, so that $\beta_L=L/L_J>1$; typically $\beta_L\approx2$ for the rf SQUID. L_J is the Josephson inductance of the junction, which is equal to $2\pi I_c/\Phi_0$. In the second type, the Persistent Current (PC) qubit, the loop is interrupted by three Josephson junctions and the amount of flux produced by the persistent circulating current is only a small fraction of the flux quantum, that is $\beta_L << 1$. Evidence for the superposition of the two macroscopic flux states have been observed in both the rf SQUID (with $\beta_L\approx2$) [4] and the PC qubit [5].

In this paper, we will focus on the effect of the inductance on the energy levels of the PC qubit. In the original description of this qubit [6], the inductance of the qubit was neglected in the calculations for the energy level. This paper includes the effects of the small inductance in the PC qubit by using a perturbative approach to quantify the initial assumptions about its energy levels.

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D. S. Crankshaw (telephone: 617-253-4699, e-mail: dscrank@mit.edu), T.P. Orlando (telephone: 617-253-5888, e-mail: orlando@mit.edu) are with the Electrical Engineering and Computer Science Department, Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

The perturbation approach simplifies the numerical calculations by reducing the dimensionality of the Schrödinger equation that must be solved. Consider a circuit with b branches, each with a Josephson junction, connected at n nodes to form m meshes (loops). In general, the dimensionality of the Schrödinger equation for such a circuit is $b=n+m-1$. If the inductance of each mesh is small so that $\beta_L<<1$, the energy levels can be calculated by ignoring the inductances (i.e., setting $\beta_L=0$). The dimensionality of the resulting Schrödinger equation is the number of independent nodes $n-1 < b$. Moreover, we find that the Hamiltonian can be written in the form

$$H_b = H_n(\Theta_n) + H_m(I_m) + \Delta H(\langle\Theta_n\rangle, \langle I_m\rangle) \,. \qquad (1)$$

The full Hamiltonian, H_b, of b variables is written in terms of three Hamiltonians: the first, $H_n(\Theta_n)$, is of the form of what one would write with $\beta_L=0$, and has n-1 node variables Θ_n [7]. $H_n(\Theta_n)$ is periodic in each of these variables. The second, $H_m(I_m)$, is of the form of a simple harmonic oscillator of the m mesh (circulating) current variables. The last term is a correction term that can often be neglected in calculating the energy levels. If we can separate the Hamiltonian this way, the mesh Hamiltonian and the correction term are easily solved analytically (since one is a simple harmonic oscillator and the other is calculated from the expectation values of the other Hamiltonians' variables), leaving only the node Hamiltonian, which has a lower dimensionality than the branch Hamiltonian and is periodic in all its variables. (This reduces the computational time of $O(N^b)$ for H_b to $O(N^{n-1})$, $O(m)$, and $O(nN+m)$ for the terms H_n, H_m, and ΔH respectively, where N is the number of discretized elements of the quantum phase variables.) We illustrate this by considering two special cases where the perturbative approach can be compared with an exact numerical calculation of the full Hamiltonian. In Section II, a single loop with one Josephson junction (the rf SQUID geometry) will be considered, and in Section III a loop with two Josephson junctions (the dc SQUID geometry). In Section IV we use this approach to calculate the effect of the inductance on the PC qubit.

II. THE RF SQUID WITH β_L<<1

The rf SQUID consists of a single Josephson junction in a superconducting loop with inductance L, as shown in Fig. 1(a). The applied flux Φ_{ext} is given in terms of the frustration, $f=\Phi_{ext}/\Phi_0$. The Hamiltonian of this system is

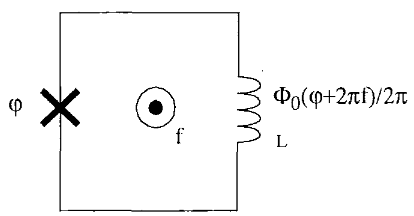
$$H_b=\frac{1}{2}C\left(\frac{\Phi_0}{2\pi}\right)^2\dot{\varphi}^2+\left(\frac{\Phi_0}{2\pi}\right)^2\frac{(\varphi+2\pi f)^2}{2L}+E_J(1-\cos\varphi), \qquad (2)$$

where $E_J=\Phi_0 I_c/2\pi$ and C is the junction capacitance. There are n=1 node variables, and m=1 mesh variables, giving a total of b=1 branch variable. This circuit is thus one-dimensional. When L=0, however, $\langle\varphi\rangle = -2\pi f$ and $H_n=E_J(1-\cos(2\pi f))$, which is zero-dimensional.

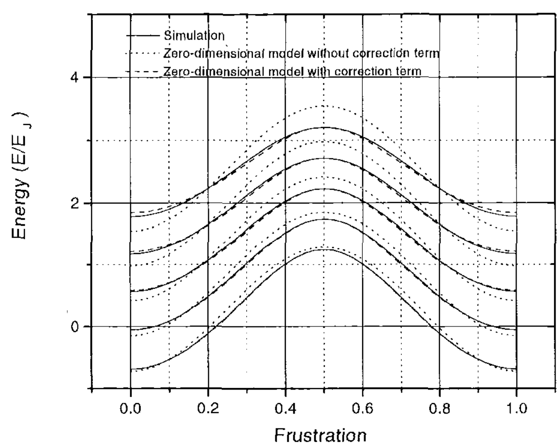
To transform (2) to the form of (1) for small inductances, the first step is to use the mesh current I, rather than phase, as the basis variable. For this transformation, $LI=\Phi_0(\varphi-2\pi f)/2\pi$, so that (2) becomes

$$H_b=E_J[1-\cos\left(2\pi LI/\Phi_0+2\pi f\right)]+\tfrac{1}{2}CL^2\dot{I}^2+\tfrac{1}{2}LI^2. \qquad (3)$$

Expanding H in terms of L, which is assumed to be small, and defining $\alpha=L_J\beta_L^2\cos(2\pi f)$, $\gamma=(E_J\beta_L/I_c)\sin(2\pi f)$, and $\tilde{I}=I+\gamma/(L+\alpha)$, we find that the Hamiltonian can be written as



(a)



(b)

Fig 1. (a) The RF SQUID circuit and (b) the energy of the lowest energy levels versus the frustration, f, for the RF SQUID where β_L=0.25.

$$H_b=\left[\tfrac{1}{2}CL^2\dot{\tilde{I}}^2+\tfrac{1}{2}(L+\alpha)\tilde{I}^2\right]+E_J[1-\cos\left(2\pi f\right)]-\frac{1}{2}\frac{\gamma^2}{L+\alpha}. \qquad (4)$$

This gives the three part Hamiltonian of (5): the first term is the β_L=0 contribution, the second term, the simple Harmonic oscillator part, and the third, a small correction term. The total energy, in terms of E_J and $E_C=e^2/2C$, is

$$\begin{aligned}E/E_J &= 1-\cos(2\pi f)\\ &\quad+(n+\tfrac{1}{2})(8E_c/E_J)^{1/2}\beta_L^{-1/2}\sqrt{1+\beta_L\cos(2\pi f)}\\ &\quad-\frac{1}{2}\frac{\beta_L\sin^2(2\pi f)}{1+\beta_L\cos(2\pi f)}.\end{aligned} \qquad (5)$$

Figure 1(b) shows a comparison of energy bands produced by (5) with those produced by the full simulation of (2), as well as the zero-dimensional solution (without the correction term). This approximation works well for the low levels for β_L<0.3.

III. DC SQUID

The DC SQUID with inductance has two independent variables corresponding to its two Josephson junctions. Its full Hamiltonian is

$$\begin{aligned}H_b &= \frac{1}{2}C\left(\frac{\Phi_0}{2\pi}\right)^2\dot{\varphi}_1^2+\frac{1}{2}C\left(\frac{\Phi_0}{2\pi}\right)^2\dot{\varphi}_2^2+\\ &\left(\frac{\Phi_0}{2\pi}\right)^2\frac{(\varphi_1-\varphi_2+2\pi f)^2}{2L}+E_J(2-\cos\varphi_1-\cos\varphi_2).\end{aligned} \qquad (6)$$

This has n=2 node variables and m=1 mesh variables, giving b=2 branch variables, producing a two-dimensional equation. When L=0, $\varphi_2=\varphi_1+2\pi f$, which reduces the Hamiltonian from two dimensions to one dimension, resulting in a periodic Hamiltonian,

$$H_n=C\left(\frac{\Phi_0}{2\pi}\right)^2\dot{\varphi}_1^2+2E_J\left(1-\cos(\pi f)\cos(\pi f-\varphi_1)\right). \qquad (7)$$

The solution to (7) can be solved numerically. We now write the full Hamiltonian, H_b, in terms of a node variable Θ and the mesh current variable I_m. We choose these variables so that there are no cross terms of the form $\dot{\Theta}\dot{I}_m$, in order to show that H_b can be written in the tripartite form of (1) with H_n given by (7). To this end, we model the SQUID as in Fig. 2(a). In this circuit description, there is a symmetric node with a phase of Θ. The mesh inductance of the loop is divided into two branches, one on each side of the node. The two variables of this circuit, rather than being the phases of the junctions, are the phase of the node and the mesh current of the loop. These are related to φ_1 and φ_2 by $I_m=\left(\Phi_0(\varphi_2-\varphi_1)/2\pi-\Phi_{ext}\right)/L_m$ and $\Theta=\varphi_1+\frac{1}{2}(2\pi/\Phi_0)(L_m I_m+\Phi_{ext})$ or $\Theta=\varphi_2-\frac{1}{2}(2\pi/\Phi_0)(L_m I_m+\Phi_{ext})$. This gives

$$H_b = \tfrac{1}{2} 2C\left(\frac{\Phi_0}{2\pi}\right)^2 \dot{\Theta}^2 - 2E_J \cos\left(\frac{2\pi}{\Phi_0}\frac{L_m I_m + \Phi_{ext}}{2}\right)\cos\Theta + \tfrac{1}{8}(2C)L_m^2 \dot{I}_m^2 + \tfrac{1}{2} L_m I_m^2 . \tag{9}$$

Assuming that L_m is small allows the expansion of the cosine term in (9) according to the parameter $L_m I_m/\Phi_0$. Additionally, the $L_m I_m \cos\Theta$ term can be replaced with $L_m I_m \langle\cos\Theta\rangle$, since this small term has little effect on the Θ-dependent portion of the Hamiltonian, but a significant effect on the $L_m I_m$ portion. This, combined with the substitution of $\tilde{I}_m = I_m + I_c \sin(\pi\Phi_{ext}/\Phi_0)\langle\cos\Theta\rangle$, which effectively zeroes the current on its expected value according to the flux through the loop, gives the Hamiltonian in (10).

$$\begin{aligned} H_b = & \left[\frac{1}{2}2C\left(\frac{\Phi_0}{2\pi}\right)^2 \dot{\Theta}^2 - 2E_J \cos\left(\frac{\pi\Phi_{ext}}{\Phi_0}\right)\cos\Theta\right] + \\ & \left[\frac{1}{8}(2C)L_m^2 \dot{\tilde{I}}_m^2 + \tfrac{1}{2} L_m \tilde{I}_m^2\right] \\ & -\frac{1}{2}L_m\left(I_c \sin\left(\frac{\pi\Phi_{ext}}{\Phi_0}\right)\langle\cos\Theta\rangle\right)^2 + \\ & E_J\left(\frac{\pi L_m}{\Phi_0}\left[\tilde{I}_m - I_c \sin\left(\frac{\pi\Phi_{ext}}{\Phi_0}\right)\langle\cos\Theta\rangle\right]\right)^2 \\ & \times\cos\left(\frac{\pi\Phi_{ext}}{\Phi_0}\right)\langle\cos\Theta\rangle \end{aligned} \tag{10}$$

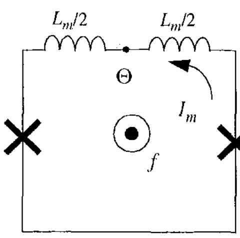
The first two terms form a Hamiltonian identical to (7) with variable Θ rather than φ_1, while the second two form a harmonic oscillator in $\tilde{I}_m$. The last term contains L_m^2, which is small and can be neglected. That leaves the fifth term, which is a classical constant and does not affect the quantum system. Thus the energy bands for this system are given by

$$E = E_{1D}(f) + \hbar\omega_0(n+\tfrac{1}{2}) - \tfrac{1}{2}L_m\left(I_c \sin\left(\frac{\pi\Phi_{ext}}{\Phi_0}\right)\langle\cos\Theta\rangle\right)^2 . \tag{11}$$

Here, E_{1D} are the energy bands from (6) and $\hbar\omega_0 = 4\left(E_J/\beta_L^{1/2}\right)\left(E_c/E_J\right)^{1/2}$. This result is plotted in Fig. 2(b), alongside the results from simulation, which are in good agreement.

IV. The Persistent Current Qubit

The PC qubit discussed by Mooij *et al.*[6] consists of three junctions: two of the junctions have identical energy E_J while the third has an energy of αE_J. A typical value of α is 0.8. This circuit has n=3 node variables and m=1 mesh variables, for a total of b=3 branch variables. This three dimensional circuit is too computationally intensive to numerically calculate directly. Therefore, we use our method to reduce the full Hamiltonian to two separate Hamiltonians, H_n and H_m, which are two- and one-dimensional respectively, and which are more easily calculable. For example, in the original description of the qubit, β_L=0, and the energy levels were calculated from the two-dimensional Hamiltonian



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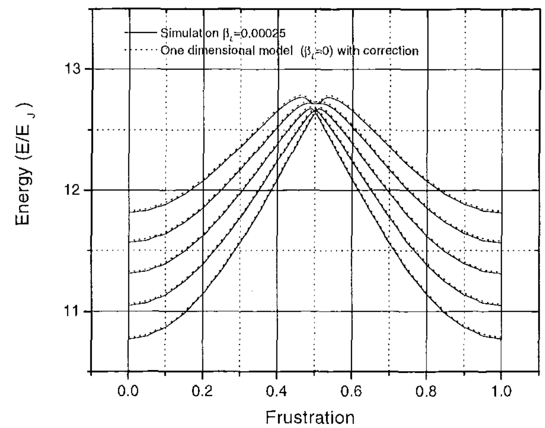

(b)

Fig. 2. (a) The circuit used to find the DC SQUID energy bands once inductance is added. (b) A comparison of simulated energy bands and analytically derived bands.

$$\begin{aligned} H_n = & \tfrac{1}{2}M_p\dot{\varphi}_p^2 + \tfrac{1}{2}M_m\dot{\varphi}_m^2 + E_J[2+\alpha-2\cos\varphi_p\cos\varphi_m - \\ & \alpha\cos(2\varphi_m + 2\pi f)] \quad , \end{aligned} \tag{12}$$

where $\varphi_p = (\varphi_1+\varphi_2)/2$ and $\varphi_m = (\varphi_1-\varphi_2)/2$.

In order to include the inductance, we need to be able to separate the mesh from the node variables so that there are no cross terms or time derivatives. In principle, this can always be done. We divide the mesh inductance into three branch inductances and the definition of two node phases of Θ_1 and Θ_2, as shown in Fig. 3(a). The initial three variables, φ_1, φ_2, and φ_3, are thus replaced:

$$\begin{aligned} \varphi_1 &= \Theta_1 - \frac{1-\xi}{2}\left(\frac{2\pi}{\Phi_0}\right)(L_m I_m + \Phi_{ext}) \\ \varphi_2 &= \Theta_2 + \frac{1-\xi}{2}\left(\frac{2\pi}{\Phi_0}\right)(L_m I_m + \Phi_{ext}) \\ \varphi_3 &= \Theta_2 - \Theta_1 - \xi\left(\frac{2\pi}{\Phi_0}\right)(L_m I_m + \Phi_{ext}) \quad . \end{aligned} \tag{13}$$

Here, $\xi = 1/(1+2\alpha)$. Next, Θ_1 and Θ_2 are replaced with $\Theta_p = (\Theta_1 + \Theta_2)/2$ and $\Theta_m = (\Theta_1 - \Theta_2)/2$, so that

$$H_b = \tfrac{1}{2}M_p\dot{\Theta}_p^2 + \tfrac{1}{2}M_m\dot{\Theta}_m^2 + \frac{\alpha}{1+2\alpha}CL_m^2\dot{I}_m^2 + E_J[2+\alpha-2\cos\Theta_p\cos\left(\Theta_m - \frac{1-\xi}{2}\left(\frac{2\pi}{\Phi_0}\right)(L_mI_m+\Phi_{ext})\right) - \alpha\cos\left(-2\Theta_m - \xi\left(\frac{2\pi}{\Phi_0}\right)(L_mI_m+\Phi_{ext})\right)] + \tfrac{1}{2}L_mI_m^2 \quad (14)$$

Here, $M_p=2(\Phi_0/2\pi)^2C$ and $M_m=(2+4\alpha)(\Phi_0/2\pi)^2C$. Next, the definition $\tilde{\Theta}_m = \Theta_m - (1-\xi)2\pi f$ is substituted into the equation, facilitating an expansion of the cosine terms. The same completing-the-square technique used for the DC SQUID can be done here, using the variable $\tilde{I}_m = I_m + I_c \frac{\alpha}{1+2\alpha}\left(\left\langle\sin\left(2\tilde{\Theta}_m + 2\pi f\right)\right\rangle - 2\left\langle\cos\Theta_p \sin\tilde{\Theta}_m\right\rangle\right)$, which gives

$$H_b = \tfrac{1}{2}M_p\dot{\Theta}_p^2 + \tfrac{1}{2}M_m\dot{\tilde{\Theta}}_m^2 + E_J[2+\alpha-2\cos\Theta_p\cos\tilde{\Theta}_m - \alpha\cos\left(2\tilde{\Theta}_m+2\pi f\right)] + \frac{\alpha}{1+2\alpha}CL_m^2\dot{\tilde{I}}_m^2 + \tfrac{1}{2}L_m\tilde{I}_m^2 - \tfrac{1}{2}L_mI_c^2\left(\frac{\alpha}{1+2\alpha}\right)^2\left(\left\langle\sin\left(2\tilde{\Theta}_m+2\pi f\right)\right\rangle - \left\langle 2\cos\Theta_p\sin\tilde{\Theta}_m\right\rangle\right)^2 . \quad (15)$$

This equation shows the original PC qubit Hamiltonian of the form H_n, an independent simple harmonic oscillator term, and a classical correction term, resulting in

$$E = E_q(f) + (n+\tfrac{1}{2})\hbar\omega_0 - \tfrac{1}{2}L_mI_c^2\left(\frac{\alpha}{1+2\alpha}\right)^2\left(\left\langle\sin\left(2\tilde{\Theta}_m+2\pi f\right)\right\rangle - \left\langle 2\cos\Theta_p\sin\tilde{\Theta}_m\right\rangle\right)^2 . \quad (16)$$

In this equation, $\hbar\omega_0 = 2\sqrt{(E_c/E_J)((1+2\alpha)/\alpha)/\beta_L}$. The correction to the energy band diagram is shown in Fig. 3(b). The effect on the shape of the curve is insignificant.

V. Conclusions

The inductance term in PC qubit can legitimately be neglected when it is small. It is possible to calculate the effect of the neglected term without using the complete Hamiltonian. This is done by using three separate energy terms: the energy bands from the lower-dimensional and periodic Hamiltonian, H_n, when β_L=0, a simple harmonic oscillator term, H_m, due to the inductance, and then a small correction term. This works well with the rf SQUID, the DC SQUID, and the PC qubit. Furthermore, this method can be used to study inductance effects on more complicated circuits.

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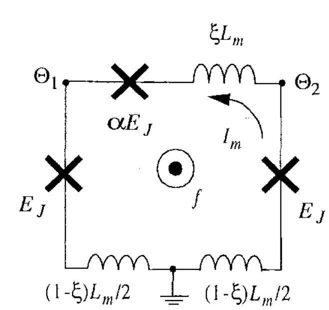

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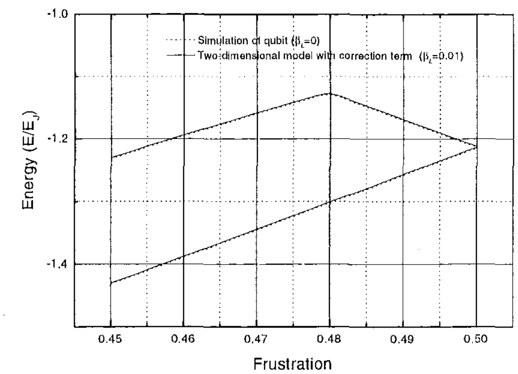

(b)

Fig. 3. (a) The circuit used to derive the PC qubit Hamiltonian with inductance. (b) A comparison of PC qubit with and without a small inductance. The β_L was chosen to reflect the estimated geometric inductance of the fabricated qubit [5]. The region used for quantum computation is shown in this diagram.

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[7] In principle, this can always be done for such circuits. A general proof can be given in terms of topological matrices. See E. Trías, to be published.