2011 Data Compression Conference

Scalar Quantization for Relative Error*

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Abstract

Quantizers for probabilistic sources are usually optimized for mean-squared error. In many applications, maintaining low relative error is a more suitable objective. This measure has previously been heuristically connected with the use of logarithmic companding in perceptual coding. We derive optimal companding quantizers for fixed rate and variable rate under high-resolution assumptions. The analysis shows logarithmic companding is optimal for variable-rate quantization but generally not for fixed-rate quantization. Naturally, the improvement in relative error from using a correctly optimized quantizer can be arbitrarily large. We extend this framework for a large class of nondifference distortions.

1 Introduction

The problem of optimizing a quantizer for a given distortion measure and probabilistic source model is well-studied, and it is surveyed in [1]. Most commonly, mean-squared error (MSE) is used both in theory and practice due to its efficacy in measuring signal fidelity and convenient mathematical properties, especially in conjunction with Gaussian random variables and linear systems. However, most practical applications require nonlinear post-processing or alternative distortion measures that make quantizers designed for MSE suboptimal. Recent work in functional scalar quantization (FSQ) addresses how to design quantizers when the computations that follow are known [2, 3]. A key result is that if the computations are nonlinear, using a quantizer accounting for functional sensitivity can be dramatically better than using an MSE-optimized one; this has been constructively demonstrated in the design of quantizers for compressed sensing [4]. Functional quantization is intimately related to the idea of "task-driven" quantization for inference problems [5, 6, 7].

In this work, we approach the design of quantizers for nondifference distortion measures using FSQ. In particular, the emphasis is on relative error, which is defined as

$$d_{\text{re}}(x, y) = \frac{(x-y)^2}{x^2}.$$

*This material is based upon work supported by the National Science Foundation under Grant No. 0729069.

1068-0314/11 $26.00
DOI 10.1109/DCC.2011.36

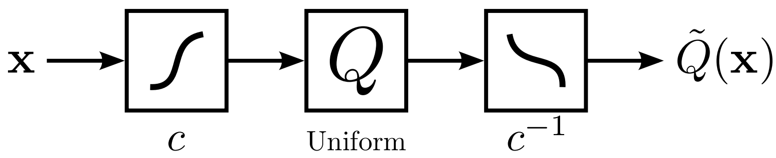

Fig. 1: A companding model for nonuniform quantization. The input $\mathbf{x}$ is transformed by a nonlinear function c and then uniformly quantized. The nonlinearity is then inverted to produce the output $\tilde{Q}(\mathbf{x})$, with tilde indicating that the quantizer is generated using the compander model.

This measure is often used explicitly in the engineering literature for numerical analysis and fidelity of floating-point representations [8]. It also serves as a heuristic justification for using logarithmic quantization [9, 10], such as μ-law and A-law companding in perceptual coding. We formalize this justification by showing logarithmic companding is indeed optimal if relative error is the measure of interest and the quantization is variable rate. We also demonstrate that this does not hold in the fixed-rate case, but the optimal quantizers may be very similar depending on the source distribution. Moreover, we present some new applications where relative error may be useful in analyzing signal acquisition.

Relative error falls under a class of nondifference distortion measures called *locally quadratic*, which has been studied both for fixed rate [11] and variable rate [12]. As such, our results may be derived using those frameworks. However, we propose an alternative derivation using FSQ, taking advantage of its intuitive construction and potential generalizations to other measures.

2 Preliminaries

We define a scalar quantizer Q_K as a mapping from the real line to a set of K points $\mathcal{C} = \{c_k\}$. In particular, the quantizer is regular if we partition $\mathbb{R}$ into a set of nonoverlapping intervals $\mathcal{P} = \{P_k\}$ and set $Q_K(x) = c_k$ if $x \in P_k$. Since the P_k's are intervals, the set $\mathcal{P}$ can also be represented by a set $\{p_k\}_{k=0}^{K}$ corresponding to the boundary points of the intervals with $p_0 = -\infty$ and $p_K = \infty$.

A widely-used mapping is uniform quantization, where the c_k's are equidistant on the support and the intervals have equal lengths (except at the ends of the support). A more general class of quantizers can be constructed using a *compander* [13] as shown in Figure 1. In this model, the source signal is transformed using a strictly increasing and smooth nonlinear function $c : \mathbb{R} \to [0, 1]$, then quantized using a uniform quantizer with K levels on the support $[0, 1]$, and finally passed through the inverse of c. The net effect is a sequence of nonuniform quantizers indexed by K and defined by a single function c. Commonly, we define a *point density function* λ to satisfy

$$\lambda(x) \propto c'(x);$$

$$\int \lambda(x)\, dx = 1,$$

which forms a one-to-one correspondence to c. One interpretation of the point density for large K is that $\lambda(x)\delta$ is the approximate fraction of codewords in an interval centered at x with width δ.

The output of a quantizer is often converted into a binary stream and described with a rate which is used as a parameter to study performance. For the case when each codeword has the same length (fixed rate), the rate is defined as $R_{\text{fr}} = \log_2(K)$. For the case when the codewords are entropy-coded based on the the distribution of $Q_K(\mathbf{x})$ (variable rate), the rate is defined as $R_{\text{vr}} = H(Q_K(\mathbf{x}))$. This is related to the number of quantization levels through the approximation $R_{\text{vr}} \approx h(\mathbf{x}) + \log_2 K + \mathbf{E}[\log_2 \lambda(\mathbf{x})]$ [14].

2.1 Quantization for MSE

Consider an iid scalar source $\mathbf{x}$ with a probability density $f_{\mathbf{x}}$ to be quantized by $\tilde{Q}_K$ constructed using the compander model given λ. The MSE is defined as

$$D_{\text{mse}}(K) = \mathbf{E}[|\mathbf{x} - \tilde{Q}_K(\mathbf{x})|^2].$$

When K is large, the distortion can be simplified using high-resolution approximations to yield

$$D_{\text{mse}}(K) \simeq \frac{1}{12K^2}\mathbf{E}[\lambda^{-2}(\mathbf{x})], \tag{1}$$

where $\simeq$ indicates that the ratio of the two expressions approaches 1 as K increases. Hence, the performance of a scalar quantizer for MSE can be determined solely by a simple relationship between the source distribution, point density and size of the codebook that becomes more precise when K is large. For a given K, the construction using the compander model does not necessarily give the best possible MSE. However, it is asymptotically optimal [15], meaning

$$\lim_{K\to\infty} \frac{\inf_{\tilde{Q}_K} \mathbf{E}[|\mathbf{x} - \tilde{Q}_K(\mathbf{x})|^2]}{\inf_{Q_K} \mathbf{E}[|\mathbf{x} - Q_K(\mathbf{x})|^2]} = 1.$$

In practice, we find that this approximation is reasonable even for moderate values of K.

For fixed-rate quantization, Hölder's inequality is used to show that the optimal point density is

$$\lambda^*_{\text{mse,fr}}(x) \propto f_{\mathbf{x}}^{1/3}(x), \tag{2}$$

and the resulting distortion is

$$D^*_{\text{mse,fr}}(R) \simeq \frac{1}{12}\|f_{\mathbf{x}}\|_{1/3} 2^{-2R}, \tag{3}$$

with the notation $\|f\|_p = (\int_{-\infty}^{\infty} f^p(x)\,dx)^{1/p}$ [16]. This is visualized in Figure 2 for a sample source distribution and K.

For variable-rate quantization, Jensen's inequality is used to show the optimal point density $\lambda^*_{\text{mse,vr}}$ is constant on the support of the input distribution [16], so the optimal quantizer is uniform and the resulting distortion is

$$D^*_{\text{mse,vr}}(R) \simeq \frac{1}{12} 2^{-2(R-h(\mathbf{x}))}. \tag{4}$$

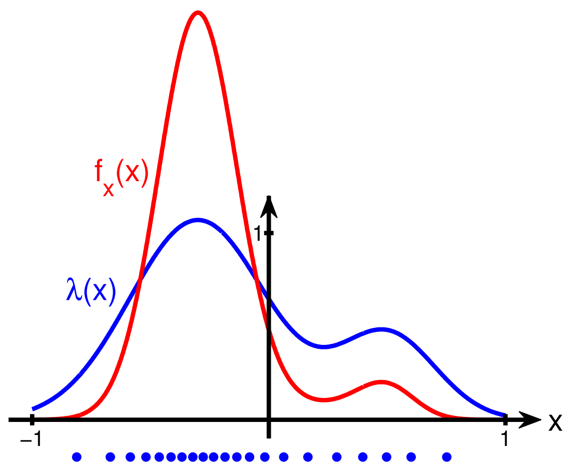

Fig. 2: An example of a source probability density function and the corresponding optimal point density function for fixed-rate quantization, defined as $\lambda^*_{\text{mse,fr}}(x) \propto f_{\mathbf{x}}^{1/3}(x)$. A quantizer with $K = 21$ is represented by the dots corresponding to the placement of codewords.

2.2 Functional Scalar Quantization

More recently, FSQ has been proposed for when the computation following quantization is known. Given that the quantizer output is transformed by a function g, the error measure is now

$$D_{\text{fsq}}(K) = \mathbf{E}[|g(\mathbf{x}) - g(\tilde{Q}_K(\mathbf{x}))|^2]. \tag{5}$$

We require $f_{\mathbf{x}}$ and g to satisfy certain smoothness conditions as discussed in [3] and define the *sensitivity profile* to be $\gamma(x) = |g'(x)|$. The distortion performance is then

$$D_{\text{fsq}}(K) \simeq \frac{1}{12K^2}\mathbf{E}\left[\left(\frac{\gamma(\mathbf{x})}{\lambda(\mathbf{x})}\right)^2\right], \tag{6}$$

which is (1) with an added factor γ.

For fixed-rate quantization, the optimal point density is

$$\lambda^*_{\text{fsq,fr}}(x) \propto \left(\gamma^2(x) f_{\mathbf{x}}(x)\right)^{1/3}, \tag{7}$$

resulting in distortion

$$D^*_{\text{fsq,fr}}(R) \simeq \frac{1}{12}\|\gamma^2 f_{\mathbf{x}}\|_{1/3} 2^{-2R}. \tag{8}$$

For variable-rate quantization, the optimal point density is

$$\lambda^*_{\text{fsq,vr}}(x) \propto \gamma(x), \tag{9}$$

resulting in distortion

$$D^*_{\text{fsq,vr}}(R) \simeq \frac{1}{12} 2^{-2(R - h(\mathbf{x}) - \mathbf{E}[\log_2 \gamma(\mathbf{x})])}. \tag{10}$$

3 Quantization for Relative Error

We now consider expected relative error, which takes the form

$$D_{\text{re}}(K) = \mathbf{E}\left[\frac{|\mathbf{x} - \tilde{Q}_K(\mathbf{x})|^2}{\mathbf{x}^2}\right]. \tag{11}$$

Relative error corresponds to the squared error scaled by the energy of the input, making the result scale invariant. In this section, we demonstrate how to optimize quantizers for relative error.

First, we find the distortion performance of a quantizer $\tilde{Q}_K$, which is analogous to (1) and (6).

Theorem 1. *Consider a memoryless source* $\mathbf{x}$ *with probability density* $f_{\mathbf{x}}$ *that is smooth on a compact subinterval of* $(0, \infty)$ *and zero elsewhere. The source is quantized using a nonuniform scalar quantizer constructed using the compander model and specified by* λ *and* K*. The relative error between the output of the quantizer and the source satisfies*

$$\lim_{K\to\infty} D_{\text{re}}(K)K^2 = \frac{1}{12}\mathbf{E}[\mathbf{x}^{-2}\lambda^{-2}(\mathbf{x})],$$

which we express also as

$$D_{\text{re}}(K) \simeq \frac{1}{12K^2}\mathbf{E}[\mathbf{x}^{-2}\lambda^{-2}(\mathbf{x})].$$

This theorem can be proven by emulating the steps for the derivation of (1) (see, e.g., [1]) with the new cost. We will instead follow the FSQ steps [3, Appendix A] to demonstrate that the results hold for a class of fidelity measures beyond relative error.

Consider a fidelity measure that takes the form

$$d(x, y) = n(x)\left(m(x) - m(y)\right)^2,$$

where m satisfies the smoothness conditions in FSQ and n is non-negative, bounded and piecewise smooth. To compute expected cost for the quantized signal, we use the total expectation theorem to separate the error terms for the quantization cells and use Taylor expansion to rewrite the difference term as

$$m(x) - m(c_k) = m'(c_k)(x - c_k) + \mathcal{O}((x - c_k)^2)$$

for $x \in P_k$. The residual terms are inconsequential and the distortion becomes

$$\begin{aligned} D_{\text{fsq}}(K, \lambda) &= \sum_{k=1}^{K} \mathbf{E}\left[n(\mathbf{x})\left(m(\mathbf{x}) - m(c_k)\right)^2 \mid \mathbf{x} \in P_k\right] \mathbf{P}(\mathbf{x} \in P_k) \\ &\overset{(a)}{\approx} \sum_{k=1}^{K} \mathbf{E}\left[n(\mathbf{x})\left(m'(c_k)(\mathbf{x} - c_k)\right)^2 \mid \mathbf{x} \in P_k\right] \mathbf{P}(\mathbf{x} \in P_k) \end{aligned}$$

$$
\begin{aligned}
&\overset{(b)}{\approx} \sum_{k=1}^{K} \mathbf{E}\left[n(c_k)\left(m'(c_k)(\mathbf{x}-c_k)\right)^2 \mid \mathbf{x} \in P_k\right] \mathbf{P}(\mathbf{x} \in P_k) \\
&\overset{(c)}{\approx} \frac{1}{12} \sum_{k=1}^{K} \left(\frac{\sqrt{n(c_k)}m'(c_k)}{K\lambda(c_k)}\right)^2 \mathbf{P}(\mathbf{x} \in P_k) \\
&\overset{(d)}{\approx} \frac{1}{12K^2} \mathbf{E}\left[\gamma^2(\mathbf{x})/\lambda^2(\mathbf{x})\right],
\end{aligned}
$$

where (a) follows from Taylor expansion, (b) holds when K is large and n is bounded and smooth, (c) uses the high-resolution approximation of $\text{length}(P_i) \approx (K\lambda(c_i))^{-1}$, and (d) follows from setting $\gamma(x) = |\sqrt{n(x)}m'(x)|$ and using the standard high-resolution technique of approximating the expectation using a Riemann sum.

For relative error, the conditions are satisfied and $\gamma(x) = 1/x$, which leads to the distortion result in the theorem. Moreover, since this expression matches (6), we can easily find the optimal point densities.

Corollary 1. *For a given source probability density $f_{\mathbf{x}}$ with support contained in $[a,b]$ with $0 < a < b < \infty$, the optimal point density for fixed-rate quantization is*

$$
\lambda^*_{\text{re,fr}}(x) = \frac{x^{-2/3} f_{\mathbf{x}}^{1/3}(x)}{\int_a^b t^{-2/3} f_{\mathbf{x}}^{1/3}(t)\, dt}, \qquad \text{if } x \in [a,b]; \quad \text{and } 0 \text{ otherwise.}
$$

Corollary 2. *For a given source probability density $f_{\mathbf{x}}$ with support contained in $[a,b]$ with $0 < a < b < \infty$, the optimal point density for variable-rate quantization is*

$$
\lambda^*_{\text{re,vr}}(x) = \frac{1/x}{\int_a^b 1/t\, dt}, \qquad \text{if } x \in [a,b]; \quad \text{and } 0 \text{ otherwise.}
$$

The variable-rate case is particularly interesting because the equivalent compander is $c(x) = \ln(x)$, meaning the codewords are uniform on a logarithmic scale over the support of the source. As for the MSE-optimized quantizer, $\lambda^*_{\text{re,vr}}$ does not depend on the source distribution except in that it spans the same support. In general, the fixed-rate quantizer will not be logarithmic except when $f_{\mathbf{x}}(x) \propto 1/x$.

4 Numerical Results

As expected, if the true cost is relative error, using λ^*_{re} is better than λ^*_{mse}. In fact, the improvements can be substantial if the source has support that spans several orders of magnitude. However, the measure is ill-posed if the support includes 0, so we currently restrict our attention to sources that take strictly positive values.

For example, consider $\mathbf{x}$ uniformly distributed on $[a,b]$ for $0 < a < b < \infty$. Using (2), the fixed-rate quantizer designed for MSE is given by

$$
\lambda^*_{\text{mse,fr}}(x) = \begin{cases} 1/(b-a), & \text{if } x \in [a,b]; \\ 0, & \text{otherwise.} \end{cases}
$$

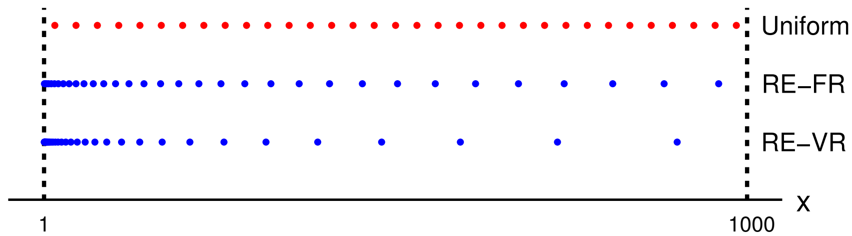

Fig. 3: Codebooks for fixed-rate and variable-rate quantizers optimized for relative error using a source uniformly distributed on $[1, 1000]$, as well as for a uniform quantizer, which would be optimal if the cost was MSE. The relative-error quantizers have finer quantization for smaller magnitude since the scaled error will be greater. The two variable-rate quantizers only depend on the support of the source, not its distribution.

Using Corollary 1, the fixed-rate quantizer designed for relative error is

$$\lambda^*_{\text{re,fr}}(x) = \begin{cases} x^{-2/3}/3(b^{1/3} - a^{1/3}), & \text{if } x \in [a, b]; \\ 0, & \text{otherwise.} \end{cases}$$

Applying Theorem 1 to both point densities then yields the performance. In particular, the best relative error for the uniform distribution is

$$D^*_{\text{re,fr}}(R) = \frac{27}{12} \cdot \frac{b^{1/3} - a^{1/3}}{b - a} 2^{-2R}. \tag{12}$$

Letting $(a, b) = (1, 10)$, the optimal relative-error quantizer yields a 2.4 dB improvement over the MSE quantizer. Meanwhile, $(a, b) = (1, 1000)$ leads to a performance improvement of 17 dB. We can see this can be arbitrarily large depending on the support of $\mathbf{x}$.

In the variable-rate case, both the MSE-optimized and RE-optimized quantizers only depend on the support of the source, and the codewords are uniformly placed on linear and logarithmic scales respectively. Again, we can find the distortion using Theorem 1, with the best possible performance for this source being

$$D^*_{\text{re,vr}}(R) = \frac{(b - a)^2}{12} 2^{-2(R-C)}, \tag{13}$$

where $C = 1/\ln(2) - (b \log_2 b - a \log_2 a)/(b - a)$. Letting $a = 1$ and $b = 10$ or 1000, $\lambda^*_{\text{re,vr}}$ will yield an additional performance gain of 1.1 and 4.3 dB respectively over $\lambda^*_{\text{re,fr}}$. These gains are for any rate since all quantizers considered have the same 2^{-2R} decay.

Figure 3 shows how the codebooks of the respective types of quantization differ and Figure 4 demonstrates the performance trends as the length of the support changes.

5 Generalized Relative Error

One major limitation of relative error is that it is not well-defined when the support of $f_{\mathbf{x}}$ includes 0. This is because the error can grow without bound when $\mathbf{x}$ is very

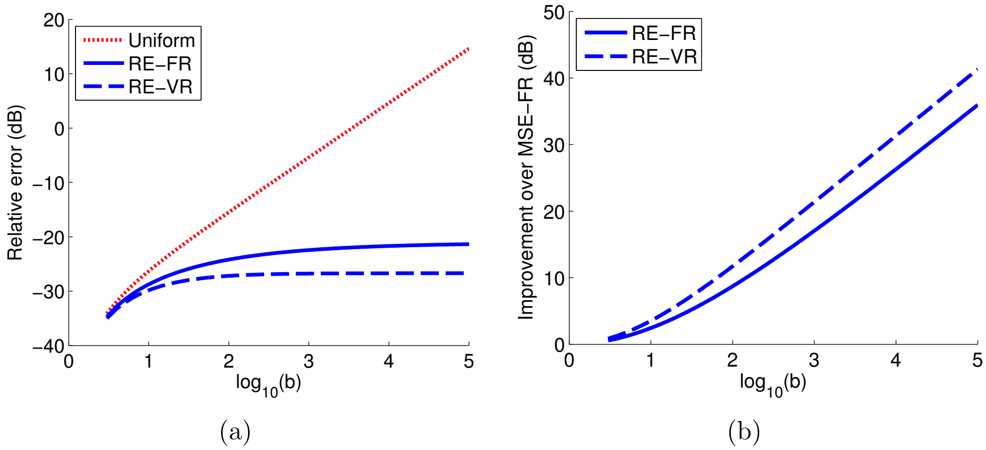

Fig. 4: Performance results of the fixed- and variable-rate quantizers with a uniform source $\mathbf{x} \sim \mathcal{U}(1, b)$. The plots provide trends in terms of (a) relative error in dB for $R = 6$; and (b) performance gain over the uniform quantizer. The performance gain does not depend on rate.

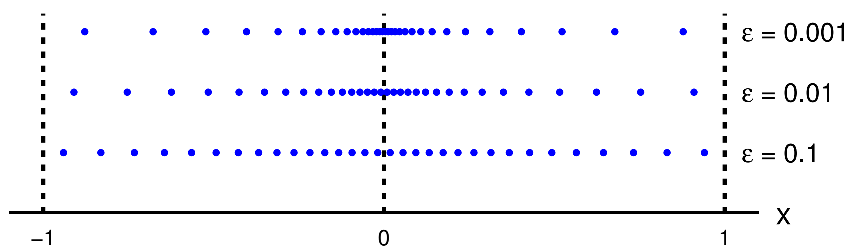

Fig. 5: Codebooks for variable-rate quantization optimized for generalized relative error when the source has support on $[-1, 1]$.

small. To combat this problem, we introduce *generalized relative error* as a way to bound the maximum relative error. Generalized relative error, parametrized by ε, is defined as

$$D_{\text{gre}}(K, \varepsilon) = \mathbf{E}\left[\frac{|\mathbf{x} - \tilde{Q}_K(\mathbf{x})|^2}{\mathbf{x}^2 + \varepsilon}\right]. \tag{14}$$

Following the steps of the proof of Theorem 1, we can show that

$$\gamma_{\text{gre}}(x, \varepsilon) = |\sqrt{n(x, \varepsilon)}m'(x)| = \frac{1}{\sqrt{x^2 + \varepsilon}}.$$

With $\gamma_{\text{gre}}(x, \varepsilon)$, we can find the optimal quantzers and their corresponding distortions. An interesting result arises for variable-rate quantization, where $\lambda^*_{\text{gre,vr}}(x, \varepsilon) \propto \gamma_{\text{gre}}(x, \varepsilon)$ yields

$$c_{\text{gre}}(x, \varepsilon) \propto \sinh^{-1}(x/\sqrt{\varepsilon}).$$

The effect of ε is demonstrated in Figure 5.

6 Extensions and Applications

As alluded to earlier, the FSQ framework can be used to find optimal quantizers for nondifference distortion measures. In the derivation of Theorem 1, we present one such class with measures that can be expressed as

$$\mathrm{Cost}(x, y) = \mathbf{E}\left[n(x)\,(m(x) - m(y))^2\right] \tag{15}$$

under some conditions on m and n. Using Taylor expansion, this class becomes the set of all input-weighted locally quadratic measures, which has been explored in [12]. Future directions of this work include generalizing the types of measures which can be represented using FSQ and exploring whether measures on vectors that may not be separable in their components fit in this framework.

We now discuss some applications where the relative error measure may lead to better performance results. In the quantization literature, relative error is usually a justification for using logarithmic companding in perceptual coding. In this work, we formalize this intuition and show logarithmic companding is indeed optimal for variable-rate quantization using the relative error measure. We also find that this is not necessarily true for fixed-rate quantization, but the optimal quantizer may have a mapping that is similar depending on the source distribution. In practice, certain parameters for μ-law and A-law companding perform well for speech, and it may be of interest to compare these quantizers to ones optimized for relative error under a realistic speech prior.

Beyond perceptual coding, relative error may have broader implications in biological systems. Using the Weber–Fechner law, we know that human perception is logarithmically proportional to many physical stimuli. Examples may be found in touch, vision [17], hearing [18], and numerical cognition [19]. However, Weber–Fechner is derived using a differential equation for which there is little neuroscientific evidence. An alternative approach to understanding this phenomenon is to model perception as quantized to a finite set of descriptions (or codewords). Hence, if these descriptions are efficient, meaning entropy-coded, and relative error is the natural measure of accuracy, then they are spaced logarithmically.

A final area of interest is signal acquisition, where relative error is relevant when gain control may be difficult to perform and the signal can have large variations. An example of this is wireless communications with fading channels [20]. If the channel gain is known, the distortion due to quantization can be greatly reduced if the gain is inverted before the ADC. However, when channel state information is not known, designing the quantizer using a relative error metric can yield better results than using an MSE quantizer.

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